202-203-204

Journal für die reine und angewandte Mathematik

gegründet von A. L. Crelle 1826

fortgeführt von

C. W. Borchardt, K. Weierstrass, L. Kronecker,
L. Fuchs, K. Hensel, L. Schlesinger

gegenwärtig herausgegeben von

Helmut Hasse und Hans Rohrbach

unter Mitwirkung von

W. Brödel, M. Deuring, A. Grothendieck, P. R. Halmos, O. Haupt,
F. Hirzebruch, E. Hopf, M. Kneser, G. Köthe, K. Prachar,
P. Roquette, W. Schmeidler, L. Schmetterer, E. Stiefel

Band 203

In 4 Heften

Mit einem Bild von A. L. Crelle

Walter de Gruyter & Co.

vormals G. J. Göschen'sche Verlagshandlung / J. Guttentag, Verlagsbuchhandlung / Georg Reimer / Karl J. Trübner / Veit & Comp.

Berlin 1960

Archiv-Nr. 180360
Satz und Druck: Walter de Gruyter & Co., Berlin W 35
Printed in Germany

Inhaltsverzeichnis des Bandes 203

Seite

Bagemihl, Frederick, Decompositions of the Plane into Three Sets 209

Biermann, Kurt-R., Urteile A. L. Crelles über seine Autoren 216

Cassels, J. W. S., Arithmetic on curves of genus 1. II. 174

Daykin, David E., Distribution of Bordered Persymmetric Matrices in a Finite Field 47

Dwork, Bernard M., On the congruence properties of the zeta function of algebraic varieties 130

Ehlich, Hartmut, Über die elementaren Beweise der Primzahlsätze 143

Ewald, Günter and **LeRoy M. Kelly,** Tangents in Real Banach Spaces. 160

Honda, Taira, On Absolute Class Fields of Certain Algebraic Number Fields . . . 80

Izbicki, Herbert, Über die explizite Automorphismengruppe von speziellen Graphen 35

Kasch, Friedrich, Ein metrischer Beitrag über Mahlersche S-Zahlen. II. 157

Kelly, LeRoy M., siehe Günter Ewald

Kneschke, A., siehe M. Schoch

McCarthy, P. J., The Generation of Arithmetical Identities. 55

Müller, Bruno, Einige Untersuchungen zur additiven Zahlentheorie auf mehrdimensionalen reellen Punktmengen 1

Nikodým, Otton Martin, Contribution to the theory of maximal, normal operators in a separable and complete Hilbert-Hermite-space. II. 90

Nohel, John A., Stability of perturbed periodic motions 64

Pinl, M., Die Hauptgruppen singulärer konformeuklidischer Räume. 40

Rieger, G. J., Einige Bemerkungen über inhomogene Linearformen 126

Schoch, M., und **A. Kneschke,** Verfahren zur Überführung gewisser differentieller Randwertprobleme in lineare Integralgleichungen 113

Thorp, Edward O., Note on Linear Operators. 110

Volkmann, Bodo, Ein metrischer Beitrag über Mahlersche S-Zahlen. I. 154

Diesem Band ist, anläßlich des 180. Geburtstages des Gründers des Journals für reine und angewandte Mathematik am 11. März 1960, ein Bild von A. L. Crelle beigegeben, das nach einem erst jetzt durch Frau I. Theurich, Berlin, bekannt gewordenen Ölgemälde im Besitz des Bezirksheimatarchivs Berlin-Schöneberg hergestellt wurde. Vgl. den Beitrag von K.-R. Biermann in diesem Band.

Ausgabedaten der Hefte von Bd. 203

Heft 1/2 ausgegeben am 15. März 1960

Heft 3/4 ausgegeben am 15. September 1960

Einige Untersuchungen zur additiven Zahlentheorie auf mehrdimensionalen reellen Punktmengen.*)

Von *Bruno Müller* in Mainz.

Einleitung

1. In der gewöhnlichen additiven Zahlentheorie untersucht man die additiven Eigenschaften der nichtnegativen ganzen Zahlen[1]). Die grundlegenden Begriffe wie Anzahlfunktion, Dichte, Summenmenge, Basis und Basisordnung wurden von L. Schnirelmann [19] eingeführt. Die beiden Hauptprobleme sind, Abschätzungen der Dichte von Summenmengen durch die Dichten der beiden Summandenmengen oder durch die Dichte der einen und die Basisordnung der andern aufzufinden.

Der erste Fall wurde durch den Satz von H. B. Mann [13] für die finite Dichte und durch den Dichtesatz von M. Kneser [10] für die asymptotische Dichte erledigt:

Theorem 1. *Sind $\mathfrak{A}$, $\mathfrak{B}$ Mengen nichtnegativer ganzer Zahlen mit den finiten Dichten α, β und der zweigliedrigen Dichte σ, und ist $0 \in \mathfrak{A} \cap \mathfrak{B}$, so gilt für die Dichte γ der Summenmenge $\mathfrak{A} + \mathfrak{B}$ die Abschätzung*

$$\gamma \geqq \min(1, \sigma) \geqq \min(1, \alpha + \beta).$$

Theorem 2. *Sind $\mathfrak{A}$, $\mathfrak{B}$ Mengen nichtnegativer ganzer Zahlen mit den asymptotischen Dichten α^*, β^* und der zweigliedrigen asymptotischen Dichte σ^*, so gilt für die asymptotische Dichte γ^* der Summenmenge $\mathfrak{A} + \mathfrak{B}$ entweder*

$$\gamma^* \geqq \sigma^* \geqq \alpha^* + \beta^*,$$

oder es gibt wenigstens eine natürliche Zahl n, so daß $\mathfrak{A} + \mathfrak{B}$ bis auf endliche viele Zahlen mit der Menge

$$(\mathfrak{A} + \mathfrak{B})^{(n)} = (x \mid x \geqq 0 \text{ und } \exists\, c \in \mathfrak{A} + \mathfrak{B} : x \equiv c \pmod{n})$$

übereinstimmt. Mit der kleinsten Zahl n_0 dieser Art ist dann

$$\gamma^* \geqq \sigma^* - \frac{1}{n_0} \geqq \alpha^* + \beta^* - \frac{1}{n_0}.$$

Für den zweiten Fall der sogenannten wesentlichen Komponenten werden die Sätze meist folgendermaßen formuliert:

---

*) Dissertation bei der naturwissenschaftlichen Fakultät der Universität Mainz (D 77); Referent Prof. Dr. H. Rohrbach. Herrn Dr. F. Kasch danke ich für seine Anregungen und Ratschläge bei der Abfassun dieser Arbeit.

[1]) Vgl. den Ergebnisbericht von H. Ostmann [15] und dessen ausführliches Literaturverzeichnis.

Theorem 3. *Sind $\mathfrak{A}$, $\mathfrak{B}$ Mengen nichtnegativer ganzer Zahlen, ist α (bzw. α^*) die (asymptotische) Dichte von $\mathfrak{A}$ und ist $\mathfrak{B}$ (asymptotische) Basis der (asymptotischen) mittleren Ordnung λ (bzw. λ^*), so gilt*

$$\gamma \geqq \alpha\left(1 + c\frac{1-\alpha}{\lambda}\right), \quad \gamma^* \geqq \alpha^*\left(1 + c^*\frac{1-\alpha^*}{\lambda^*}\right).$$

Das erste Ergebnis erzielten P. Erdös [3] und E. Landau [12], die $c = \frac{1}{2}$ zeigten. $\left(\text{H. Rohrbach [17]: } c^* = \frac{1}{2}\right)$. Die besten zur Zeit bekannten Abschätzungen stammen von F. Kasch [5], [6]. Dabei ist c eine komplizierte Funktion von α, λ und γ (bzw. α^*, λ^*, γ^*) mit Werten in der Nähe von 1, wobei diese Zahl insbesondere für kleine α, λ übertroffen wird.

2. Verschiedene Autoren haben diese Begriffsbildungen, Fragestellungen und Methoden auf andere Grundbereiche an Stelle der Menge der nichtnegativen ganzen Zahlen übertragen.

Die Menge der nichtnegativen reell n Zahlen wurde bereits von Schnirelmann [20] untersucht. Dabei benutzte er an Stelle der Anzahlfunktion das innere lebesguesche Maß. Es gelang ihm, sowie D. Raikov [16] und H. Schell [18], Analoga zu den obigen Ergebnissen der gewöhnlichen Zahlentheorie zu beweisen. Insbesondere gilt nach Schell die Abschätzung des Mannschen Satzes (Theorem 1) hier sowohl für die finite als auch für die asymptotische Dichte — für letztere also ohne den Ausnahmefall von Theorem 2. Bei der Übertragung von Theorem 3 bemerkte Raikov, daß man hier nicht die volle Basiseigenschaft von $\mathfrak{B}$ vorauszusetzen braucht. Er führte daher den neuen Begriff der schwachen Basis ein — das ist eine Menge, die nicht mehr alle nichtnegativen reellen Zahlen, sondern nur eine überall dichte Teilmenge derselben additiv darzustellen gestattet — und bewies damit

Theorem 4. *Sind $\mathfrak{A}$, $\mathfrak{B}$ Mengen nichtnegativer reeller Zahlen, ist α die Dichte von $\mathfrak{A}$ und ist $\mathfrak{B}$ schwache Basis der mittleren schwachen Ordnung λ, so gilt*

$$\gamma \geqq \alpha\left(1 + c\frac{1-\alpha}{\lambda}\right).$$

Raikov selbst zeigte $c = \frac{1}{2}$, und Schell wies nach, daß für c wie auch für c^* im asymptotischen Fall dieselben Werte, die Kasch in der gewöhnlichen Zahlentheorie hergeleitet hatte, zulässig sind.

Mengen von r-Tupeln nichtnegativer ganzer Zahlen, sog. *r-dimensionale Gitterpunkte*, wurden von L. Cheo [2] und Kasch [7], [8] und Mengen von r-Tupeln nichtnegativer reeller Zahlen, sog. *r-dimensionale Punktmengen*, von L. Chatrovsky [1] betrachtet. Anordnung und Addition werden koordinatenweise erklärt; insbesondere heißt ein Punkt dann größer als ein anderer, wenn alle seine Koordinaten größer als die entsprechenden des andern sind. Offensichtlich ist diese Anordnung für $r > 1$ nicht mehr total. Aus diesem Grund kann man zahlreiche der im eindimensionalen Fall benutzten Beweismethoden, vor allem die e-Transformation zum Beweis des Mannschen Satzes, nicht übernehmen; deswegen ist es bisher auch nicht gelungen, Analoga zum Mannschen Satz herzuleiten. Die einzigen bekannten Abschätzungen der Dichte von Summenmengen durch die Dichten der beiden Summandenmengen sind die folgenden Sätze über die finite Dichte von Gitterpunktmengen:

Theorem 5 (Cheo [2]). *Sind $\mathfrak{A}$, $\mathfrak{B}$ zweidimensionale Gitterpunktmengen mit $(0, 0) \in \mathfrak{A} \cap \mathfrak{B}$ und enthält $\mathfrak{B}$ alle Punkte einer Koordinatenhalbachse, so gilt*

$$\gamma \geqq \alpha + \beta - \alpha\beta.$$

Theorem 6 (Kasch [7]). *Sind $\mathfrak{A}$, $\mathfrak{B}$ beliebige zweidimensionale Gitterpunktmengen mit $(0, 0) \in \mathfrak{B}$, so gilt*

$$\gamma \geqq \alpha + \frac{1}{2}\alpha(1 - \alpha)\beta.$$

Theorem 7 (Kasch [7]). *Sind $\mathfrak{A}$, $\mathfrak{B}$ beliebige r-dimensionale Gitterpunktmengen mit $(0, \ldots, 0) \in \mathfrak{B}$, so gilt*

$$\gamma \geqq \alpha + \frac{1}{2r(3(r+1))^{r+1}}\alpha(1 - \alpha)\beta.$$

Weitere und bessere Abschätzungen dieser Art scheinen nur sehr schwer zu erreichen zu sein; insbesondere ist die genaue Übertragung des Mannschen Satzes für Gitterpunktmengen überhaupt nicht möglich, wie Cheo [2] durch ein Gegenbeispiel gezeigt hat.

Erfolgreicher war man bei der Übertragung der Abschätzungen für wesentliche Komponenten, bei deren Beweis die Anordnung keine so entscheidende Rolle spielt. Für Gitterpunktmengen zeigte Kasch [6] durch Nachbildung der eindimensionalen Schlußweise

Theorem 8. *Sind $\mathfrak{A}$, $\mathfrak{B}$ r-dimensionale Gitterpunktmengen, ist α die Dichte von $\mathfrak{A}$ und ist $\mathfrak{B}$ Basis mit endlicher mittlerer Ordnung λ, so gilt*

$$\gamma \geqq \alpha\left(1 + \frac{1 - 2^{r-1}\alpha}{2\lambda^r}\right).$$

Entsprechendes leitete er für die asymptotischen Größen her. Diese Ungleichungen sind allerdings nur für $0 < \alpha$, $\alpha^* < \frac{1}{2^{r-1}}$ nichttrivial. Für alle $0 < \alpha$, $\alpha^* < 1$ brauchbar dagegen ist Kaschs zweites Ergebnis:

Theorem 9. *Unter den Voraussetzungen von Theorem 8 gilt auch*

$$\gamma \geqq \alpha\left(1 + c\frac{1 - \alpha}{\lambda}\right)$$

und Entsprechendes für die asymptotischen Größen.

Im einzelnen zeigte Kasch in [7], daß $c = c^* = \frac{1}{(3(r+1))^{r+1}}$, und in [8], daß sogar

$$c = \sup_{q>1}\left(\frac{1}{(r+1)^{r+1}q^r}\min_{k=1,\ldots,r}((q-1)^k(r+1-k))\right) \text{ und } c^* = \frac{1}{(r+1)^{r+1}}$$

zulässig sind.

Für reelle Punktmengen hatte bereits Chatrovsky [1] das Analogon zu Theorem 8 bewiesen, wobei für $\mathfrak{B}$ wieder nur die schwache Basiseigenschaft vorausgesetzt wird und λ die schwache mittlere Ordnung bedeutet. Kasch vermerkte ohne Beweis, daß man durch Benutzung seines Gedankenganges in [7] auch Theorem 9 auf reelle Punktmengen übertragen könne.

3. *Ziel der vorliegenden Arbeit* ist, die Untersuchungen zur additiven Zahlentheorie auf r-dimensionalen reellen Punktmengen weiter voranzutreiben.

Im ersten Kapitel werden die beiden Basisbegriffe der gewöhnlichen und der schwachen Basis und die verschiedenen Basisordnungen betrachtet, wobei ich die Definitionen in den §§ 1, 2 im wesentlichen von Chatrovsky übernehme. In § 2 wird gezeigt, daß sich die schwachen Basen auf die gewöhnlichen abgeschlossenen Basen zurückführen lassen, indem nämlich die Klasse aller schwachen Basen bestimmter schwacher Ordnungen gerade die Klasse aller überall dichten Teilmengen von abgeschlossenen gewöhnlichen Basen derselben gewöhnlichen Ordnungen ist (Satz 2. 4). Weiter werden mehrere Ungleichungen zwischen den verschiedenen Ordnungen derselben Menge hergeleitet (Sätze 2. 2 und 2. 5), die Analoga in der gewöhnlichen Zahlentheorie haben. Schließlich wird in § 3 gezeigt, daß jede Menge positiver finiter Dichte Basis (endlicher Ordnung) und jede Menge, deren abgeschlossene Hülle positive finite Dichte hat, schwache Basis (endlicher schwacher Ordnung) ist, und es werden Abschätzungen der Basisordnungen durch die Dichten angegeben. Entsprechende Aussagen über asymptotische Basen und Dichten werden nur für den Fall, daß die Dichten sogar $> \frac{1}{2}$ sind, abgeleitet.

Im zweiten Kapitel werden für wesentliche Komponenten, für die bisher ja nur das Analogon zu Theorem 8 von Chatrovsky bewiesen und das zu Theorem 9 von Kasch ohne Beweis angegeben worden ist, folgendes gezeigt:

Satz. *Sind $\mathfrak{A}$, $\mathfrak{B}$ r-dimensionale reelle Punktmengen, hat $\mathfrak{A}$ die Dichte α (bzw. α^*) und ist $\mathfrak{B}$ schwache (asymptotische) Basis der (asymptotischen) schwachen mittleren Ordnung λ (bzw. λ^*), so gelten für die Dichten γ, γ^* von $\mathfrak{A} + \mathfrak{B}$ die Ungleichungen*

$$\gamma \geqq \alpha\left(1 + c\frac{1-\alpha}{\lambda}\right), \quad \gamma^* \geqq \alpha^*\left(1 + c^*\frac{1-\alpha^*}{\lambda^*}\right),$$

wobei $c = c^* = F(r)$ den Abschätzungen

$$\frac{1}{2}\left(\frac{2w_0}{r+w_0}\right)^{r+1} \leqq F(r) \leqq \frac{1}{\sqrt{2}}\left(\frac{2w_0}{r+v_0}\right)^{r+1}$$

mit $w_0 = \ln(1+\sqrt{2}) = 0{,}8814\ldots$ und $v_0 = \frac{1}{2\sqrt{2}}(2 - w_0) = 0{,}3955\ldots$ genügt.

Ferner wird das asymptotische Verhalten von $F(r)$ für $r \to \infty$ zu

$$F(r) = \frac{1}{\sqrt{2}}\, e^{\frac{w_0}{2\sqrt{2}}-1}\left(\frac{2w_0}{r}\right)^{r+1}\left(1 + O\left(\frac{1}{r}\right)\right)$$

bestimmt. Diese Größenordnung ist günstiger als das von Kasch vermutete $\frac{1}{(r+1)^{r+1}}$. Für $r = 1$ ist $F(r) = \frac{1}{2}$. Unter der zusätzlichen Voraussetzung, daß die natürliche Dichte von $\mathfrak{A}$ existiert, kann $c^* = \frac{1}{2^r}$ gesetzt werden, was noch erheblich besser als $c^* = F(r)$ ist.

Der Beweis dieser Resultate läuft in seinem ersten Teil den Überlegungen von Kasch und Chatrovsky bei der Herleitung des Theorems 8 weitgehend parallel. Ich übertrage insbesondere die Umformung, die Kasch bei den Gitterpunktmengen den Übergang zu Theorem 9 erlaubte. Die Verbesserung gegenüber den von Kasch für c

und c^* vermuteten Werten gelingt durch sorgfältige Abschätzung des schließlich auftretenden Integrals. Zur Bestimmung des asymptotischen Verhaltens von $F(r)$ werden einige funktionentheoretische Hilfsmittel benutzt.

I. Kapitel: Basen.

§ 1. Bezeichnungen. Zwei Hilfssätze.

4. $\mathfrak{R}$ ist die Menge aller r-Tupel reeller Zahlen $\mathfrak{x} = (x_1, \ldots, x_r)$. Zu zwei Teilmengen $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{R}$ sind $\mathfrak{A} - \mathfrak{B} = (\mathfrak{x} \mid \mathfrak{x} \in \mathfrak{A} \wedge \mathfrak{x} \in \mathfrak{B})$ die *Differenz*, $\mathfrak{A} \triangle \mathfrak{B} = (\mathfrak{A} - \mathfrak{B}) \cup (\mathfrak{B} - \mathfrak{A})$ die *symmetrische Differenz* und $\tilde{\mathfrak{A}}, \tilde{\mathfrak{B}}$ die *abgeschlossenen Hüllen*. Die Differenz ist also auch für den Fall $\mathfrak{B} \subset \mathfrak{A}$ erklärt, und es gilt dann $\mathfrak{A} - \mathfrak{B} = \mathfrak{A} - (\mathfrak{A} \cap \mathfrak{B})$. Die leere Menge wird mit $\emptyset$ und der Punkt $(0, \ldots, 0)$ mit $\mathfrak{o}$ bezeichnet.

Die *Summe zweier Punkte* $\mathfrak{a}, \mathfrak{b} \in \mathfrak{R}$ wird durch $\mathfrak{a} + \mathfrak{b} = (a_1 + b_1, \ldots, a_r + b_r)$ und die *Summe zweier Mengen* durch $\mathfrak{A} + \mathfrak{B} = (\mathfrak{a} + \mathfrak{b} \mid \mathfrak{a} \in \mathfrak{A}, \mathfrak{b} \in \mathfrak{B})$ definiert. Zur Abkürzung setzt man $\mathfrak{A} + \cdots + \mathfrak{A} = k\mathfrak{A}$.

$\mathfrak{a} < \mathfrak{b}$ bzw. $\mathfrak{a} \leqq \mathfrak{b}$ bedeutet $a_i < b_i$ bzw. $a_i \leqq b_i$ $(i = 1, \ldots, r)$. Diese Festsetzung vermittelt eine *teilweise Anordnung* von $\mathfrak{R}$, die genau für $r = 1$ eine totale ist. Die Mengen $(\mathfrak{x} \mid \mathfrak{a} \leqq \mathfrak{x} \leqq \mathfrak{b})$, $(\mathfrak{x} \mid \mathfrak{a} \leqq \mathfrak{x} < \mathfrak{b})$, $(\mathfrak{x} \mid \mathfrak{a} < \mathfrak{x} < \mathfrak{b})$ werden durch $[\mathfrak{a}, \mathfrak{b}]$, $[\mathfrak{a}, \mathfrak{b})$, $(\mathfrak{a}, \mathfrak{b})$ abgekürzt, ebenso $(\mathfrak{x} \mid \mathfrak{a} \leqq \mathfrak{x})$ und $(\mathfrak{x} \mid \mathfrak{a} < \mathfrak{x})$ durch $[\mathfrak{a}, \infty)$ und $(\mathfrak{a}, \infty)$. Diese Mengen, die ihrer Struktur nach r-dimensionale Quader sind, werden in Analogie zum eindimensionalen Fall Intervalle genannt.

Für die Mengen $[\mathfrak{o}, \infty)$ und $(\mathfrak{o}, \infty)$ benutze ich die Bezeichnung $\mathfrak{N}$ und $\mathfrak{M}$. Die additive Zahlentheorie beschäftigt sich nur mit $\mathfrak{N}$ und ihren Teilmengen. Deshalb sei für alle auftretenden Mengen $\mathfrak{A}$ automatisch $\mathfrak{A} \subset \mathfrak{N}$ vorausgesetzt. Aus $\mathfrak{A}, \mathfrak{B} \subset \mathfrak{N}$ folgt $\mathfrak{A} + \mathfrak{B} \subset \mathfrak{N}$. Für die Differenz $\mathfrak{N} - \mathfrak{A}$ wird $\bar{\mathfrak{A}}$ geschrieben.

$\chi(\mathfrak{A}, \mathfrak{a})$ ist die *charakteristische Funktion* der Menge $\mathfrak{A}$, für die also $\chi(\mathfrak{A}, \mathfrak{a}) = 1$ für $\mathfrak{a} \in \mathfrak{A}$ und $\chi(\mathfrak{A}, \mathfrak{a}) = 0$ für $\mathfrak{a} \notin \mathfrak{A}$ ist. Für alle $\mathfrak{a} \in \mathfrak{N}$ gilt $\chi(\mathfrak{A}, \mathfrak{a}) + \chi(\bar{\mathfrak{A}}, \mathfrak{a}) = 1$.

$m(\mathfrak{A})$, $\underline{m}(\mathfrak{A})$, $\overline{m}(\mathfrak{A})$ bezeichnet das (*innere*, *äußere*) *lebesguesche Maß*. $m(\mathfrak{A} \cap [\mathfrak{o}, \mathfrak{x}])$ wird durch $m(\mathfrak{A}, \mathfrak{x})$ und $m([\mathfrak{o}, \mathfrak{x}])$ durch $m(\mathfrak{x})$ abgekürzt. Das *lebesguesche Integral* einer Funktion $f(\mathfrak{x})$ über einer Menge $\mathfrak{A}$ wird mit $\int\limits_{\mathfrak{A}} f(\mathfrak{a})\, d\mathfrak{a}$ bezeichnet.

Endlich bedeutet die Schreibweise $\mathfrak{x} \to \infty$, daß alle Koordinaten des Punktes $\mathfrak{x}$ gegen ∞ gehen, $\mathfrak{x}$ also schließlich größer als jedes vorgegebene $\mathfrak{a}$ wird. Für eine beliebige Funktion $f(\mathfrak{x})$ sind $\underline{A} = \varliminf\limits_{\mathfrak{x} \to \infty} f(\mathfrak{x})$ und $\bar{A} = \varlimsup\limits_{\mathfrak{x} \to \infty} f(\mathfrak{x})$ die Zahlen

$$\underline{A} = \sup\,(\underline{a} \mid \forall \varepsilon > 0\, \exists\, \underline{\mathfrak{r}}(\varepsilon)\, \forall\, \mathfrak{x} > \underline{\mathfrak{r}}(\varepsilon) : f(\mathfrak{x}) \geqq \underline{a} - \varepsilon)$$
$$\bar{A} = \inf\,(\bar{a} \mid \forall \varepsilon > 0\, \exists\, \bar{\mathfrak{r}}(\varepsilon)\, \forall\, \mathfrak{x} > \bar{\mathfrak{r}}(\varepsilon) : f(\mathfrak{x}) \leqq \bar{a} + \varepsilon)$$

Zu jedem $\varepsilon > 0$ gibt es also ein $\mathfrak{r}(\varepsilon)$, so daß für alle $\mathfrak{x} > \mathfrak{r}(\varepsilon)$ die Ungleichung $\underline{A} - \varepsilon \leqq f(\mathfrak{x}) \leqq \bar{A} + \varepsilon$ gilt, und $\underline{A}, \bar{A}$ sind die engsten Schranken dieser Art. Ist $\underline{A} = \bar{A} = A$, so schreibt man $A = \lim\limits_{\mathfrak{x} \to \infty} f(\mathfrak{x})$.

Jedem k-gliedrigen Mengensystem $(\mathfrak{A}_1, \ldots, \mathfrak{A}_k)$, $k = 1, 2, \ldots$ werden zwei Dichten zugeordnet, die *k-gliedrige finite Dichte*

$$\sigma(\mathfrak{A}_1, \ldots, \mathfrak{A}_k) = \inf_{\mathfrak{x} > \mathfrak{o}} \frac{\sum\limits_{i=1}^{k} \underline{m}(\mathfrak{A}_i, \mathfrak{x})}{m(\mathfrak{x})} \tag{1.1}$$

und die *k-gliedrige asymptotische Dichte*

$$\sigma^*(\mathfrak{A}_1, \ldots, \mathfrak{A}_k) = \varliminf_{\mathfrak{x}\to\infty} \frac{\sum_{i=1}^{k} \underline{m}(\mathfrak{A}_i, \mathfrak{x})}{m(\mathfrak{x})}. \tag{1.2}$$

Existiert $\lim_{\mathfrak{x}\to\infty} \frac{\sum_{i=1}^{k} \underline{m}(\mathfrak{A}_i, \mathfrak{x})}{m(\mathfrak{x})}$, so heißt σ^* auch *natürliche Dichte*. Es gilt

$$0 \leqq \sigma \leqq \sigma^* \leqq k. \tag{1.3}$$

Man überlegt sich leicht, daß σ^* in folgendem Sinne translationsinvariant ist: Setzt man mit beliebigen Punkten $\mathfrak{t}_i \in \mathfrak{R}$

$$\mathfrak{A}_i' = (\mathfrak{A}_i + \{\mathfrak{t}_i\}) \cap \mathfrak{R}, \quad i = 1, \ldots, k,$$

so ist

$$\sigma^*(\mathfrak{A}_1', \ldots, \mathfrak{A}_k') = \sigma^*(\mathfrak{A}_1, \ldots, \mathfrak{A}_k).$$

Für eingliedrige Mengensysteme benutzt man speziell die Bezeichnungen

$$\begin{aligned} \sigma(\mathfrak{A}) &= \delta(\mathfrak{A}) = \alpha, \\ \sigma^*(\mathfrak{A}) &= \delta^*(\mathfrak{A}) = \alpha^*. \end{aligned} \tag{1.4}$$

5. Hilfssatz 1. 1. *Zu jeder Menge $\mathfrak{A}$ gibt es eine meßbare Teilmenge $\mathfrak{K} \subset \mathfrak{A}$, so daß $\underline{m}(\mathfrak{A}, \mathfrak{x}) = m(\mathfrak{K}, \mathfrak{x})$ gilt für alle $\mathfrak{x} \in \mathfrak{R}$. (Es bleiben also alle mit der Menge $\mathfrak{A}$ gebildeten Dichten unverändert, wenn man $\mathfrak{A}$ durch $\mathfrak{K}$ ersetzt.)*

Beweis. Nach der Maßtheorie[2]) gibt es zu jeder Menge $\mathfrak{H}$ meßbare Kerne $\mathfrak{H}^* \subset \mathfrak{H}$ mit $m(\mathfrak{H}^*) = \underline{m}(\mathfrak{H})$. Nun betrachte man die Mengen $\mathfrak{H} = \mathfrak{A} \cap [\mathfrak{o}, \mathfrak{x}]$ und zugehörige Kerne $\mathfrak{H}^* = \mathfrak{K}_\mathfrak{x}$. Dann ist

$$\mathfrak{K} = \bigcup_{\mathfrak{x} \text{ rational}} \mathfrak{K}_\mathfrak{x}$$

als abzählbare Vereinigung meßbarer Mengen meßbar, und es gilt $\mathfrak{K} \subset \mathfrak{A}$ sowie $\underline{m}(\mathfrak{A}, \mathfrak{x}) = m(\mathfrak{K}_\mathfrak{x}) \leqq m(\mathfrak{K}, \mathfrak{x}) \leqq \underline{m}(\mathfrak{A}, \mathfrak{x})$. Also ist $m(\mathfrak{K}, \mathfrak{x}) = \underline{m}(\mathfrak{A}, \mathfrak{x})$ zunächst für alle rationalen $\mathfrak{x}$, dann aber aus Stetigkeitsgründen für alle $\mathfrak{x} \in \mathfrak{R}$.

Hilfssatz 1. 2. *Sind $\mathfrak{A}$ eine meßbare und $\mathfrak{B}$ eine beliebige Menge, sowie*

$$\mathfrak{B}^* = (\mathfrak{b}^{(1)}, \mathfrak{b}^{(2)}, \ldots) \subset \widetilde{\mathfrak{B}}$$

eine abzählbare Menge von Häufungspunkten von $\mathfrak{B}$, so bleiben die Dichten von $\mathfrak{A} + \mathfrak{B}$ ungeändert, wenn man $\mathfrak{B}$ durch $\mathfrak{B}' = \mathfrak{B} \cup \mathfrak{B}^*$ oder $\mathfrak{B}'' = \mathfrak{B} - \mathfrak{B}^*$ ersetzt.

Beweis. Es genügt zu zeigen, daß $\underline{m}(\mathfrak{A} + \mathfrak{B}, \mathfrak{x})$ für alle $\mathfrak{x} \in \mathfrak{R}$ nicht größer wird, wenn $\mathfrak{B}$ durch $\mathfrak{B}'$ ersetzt wird. Nun ist[3])

$$\begin{aligned} \underline{m}(\mathfrak{A} + \mathfrak{B}', \mathfrak{x}) &\leqq \underline{m}(\mathfrak{A} + \mathfrak{B}, \mathfrak{x}) + \overline{m}((\mathfrak{A} + \mathfrak{B}') - (\mathfrak{A} + \mathfrak{B}), \mathfrak{x}) \\ &\leqq \underline{m}(\mathfrak{A} + \mathfrak{B}, \mathfrak{x}) + \sum_{i=1}^{\infty} \overline{m}((\mathfrak{A} + \{\mathfrak{b}^{(i)}\}) - (\mathfrak{A} + \mathfrak{B}), \mathfrak{x}), \end{aligned}$$

so daß es hinreicht, für alle $\mathfrak{b} \in \widetilde{\mathfrak{B}}$

$$\overline{m}((\mathfrak{A} + \{\mathfrak{b}\}) - (\mathfrak{A} + \mathfrak{B}), \mathfrak{x}) = 0$$

zu zeigen.

[2]) Vgl. P. Halmos [4], § 14.
[3]) Vgl. Halmos [4], §§ 14, 10.

Ist $\mathfrak{b} \in \widetilde{\mathfrak{B}}$, so gibt es in $\mathfrak{B}$ eine Folge $\mathfrak{b}_n \to \mathfrak{b}$, die man auch $\mathfrak{b}_n = \mathfrak{b} + \mathfrak{a}_n$, $\mathfrak{a}_n \in \mathfrak{R}$, $\mathfrak{a}_n \to \mathfrak{o}$ schreiben kann. Dann ist

$$\begin{aligned}(\mathfrak{A} + \{\mathfrak{b}\}) - (\mathfrak{A} + \mathfrak{B}) &\subset (\mathfrak{A} + \{\mathfrak{b}\}) - (\mathfrak{A} + \{\mathfrak{b} + \mathfrak{a}_1, \mathfrak{b} + \mathfrak{a}_2, \ldots\}) \\ &= (\mathfrak{A} + \{\mathfrak{b}\}) - (\mathfrak{A} + \{\mathfrak{b}\} + \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots\}),\end{aligned}$$

und mit $\mathfrak{A} + \{\mathfrak{b}\} = \mathfrak{A}'$ reduziert sich die Behauptung auf

$$\overline{m}(\mathfrak{A}' - (\mathfrak{A}' + \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots\}), \mathfrak{x}) = 0.$$

Es sei nun $\varepsilon > 0$ beliebig vorgegeben. Dann gibt es eine abzählbare Intervallüberdeckung $\mathfrak{U} = \bigcup_{i=1}^{\infty} \mathfrak{U}_i$ zu $\mathfrak{A}' \cap [\mathfrak{o}, \mathfrak{x}]$ mit

$$\sum_{i=1}^{\infty} m(\mathfrak{U}_i) - \varepsilon \leqq m(\mathfrak{A}', \mathfrak{x})$$

und ein k mit

$$\sum_{i=k+1}^{\infty} m(\mathfrak{U}_i) \leqq \varepsilon.$$

Die $\mathfrak{U}_1, \ldots, \mathfrak{U}_k$ werden so zu neuen Intervallen $\mathfrak{V}_1, \ldots, \mathfrak{V}_k$ vergrößert, daß $\mathfrak{U}_i$ in $\mathfrak{V}_i$ liegt, das Maß insgesamt um ε zunimmt und $\bigcup_{i=1}^{k} \mathfrak{U}_i$ einen endlichen Abstand $\varrho(\varepsilon)$ vom Rand von $\bigcup_{i=1}^{k} \mathfrak{V}_i$ hat. Setzt man noch $\mathfrak{A}'' = \mathfrak{A}' \cap \bigcup_{i=1}^{k} \mathfrak{U}_i$, so wird

$$m\left(\bigcup_{i=1}^{k} \mathfrak{V}_i\right) - 3\varepsilon \leqq m(\mathfrak{A}'').$$

(Mit $\mathfrak{A}$ sind auch $\mathfrak{A}'$ und $\mathfrak{A}''$ meßbar.)

Wird weiter N so groß gewählt, daß $|\mathfrak{a}_N| \leqq \varrho(\varepsilon)$ ist, so folgt aus $\mathfrak{A}'' \subset \bigcup_{i=1}^{k} \mathfrak{U}_i$

$$\mathfrak{A}'' + \{\mathfrak{a}_N\} \subset \bigcup_{i=1}^{k} \mathfrak{V}_i$$

und hieraus wegen

$$\begin{aligned}&\{\mathfrak{A}' - (\mathfrak{A}' + \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots\})\} \cap [\mathfrak{o}, \mathfrak{x}] \subset \{\mathfrak{A}' - (\mathfrak{A}'' + \{\mathfrak{a}_N\})\} \cap [\mathfrak{o}, \mathfrak{x}] \\ &\overline{\subset} \{((\mathfrak{A}' - \mathfrak{A}'') - (\mathfrak{A}'' + \{\mathfrak{a}_N\})) \cap [\mathfrak{o}, \mathfrak{x}]\} \cup (\mathfrak{A}'' - (\mathfrak{A}'' + \{\mathfrak{a}_N\})) \\ &\subset \left(\bigcup_{i=k+1}^{\infty} \mathfrak{U}_i\right) \cup \left(\bigcup_{i=1}^{k} \mathfrak{V}_i - (\mathfrak{A}'' + \{\mathfrak{a}_N\})\right)\end{aligned}$$

die Behauptung

$$\overline{m}(\mathfrak{A}' - (\mathfrak{A}' + \{\mathfrak{a}_1, \mathfrak{a}_2, \ldots\}), \mathfrak{x}) \leqq \varepsilon + m\left(\bigcup_{i=1}^{k} \mathfrak{V}_i\right) - m(\mathfrak{A}'') \leqq 4\varepsilon.$$

Bemerkung. Ob die Voraussetzung, daß $\mathfrak{A}$ meßbar sei, notwendig ist, ist mir nicht bekannt.

Auf die Voraussetzung der Abzählbarkeit von $\mathfrak{B}^*$ kann jedoch nicht verzichtet werden. Es ist nämlich leicht, Beispiele anzugeben, bei denen die Hinzunahme einer überabzählbaren Menge $\mathfrak{B}^*$ von Häufungspunkten zu $\mathfrak{B}$ die Dichte von $\mathfrak{A} + \mathfrak{B}$ vergrößert. Man wähle etwa für $\mathfrak{A}$ die Menge der Gitterpunkte in $\mathfrak{R}$, d. h. der Punkte mit nichtnegativen ganzzahligen Koordinaten, für $\mathfrak{B}$ die Menge der rationalen Punkte in $\mathfrak{R}$ — dann ist $\widetilde{\mathfrak{B}} = \mathfrak{R}$ — und für $\mathfrak{B}^*$ das Einheitsintervall $[\mathfrak{o}, \mathfrak{e}]$, $\mathfrak{e} = (1, \ldots, 1)$. Hiermit ist

einerseits $\mathfrak{A} + \mathfrak{B} = \mathfrak{B}$, also $\delta(\mathfrak{A} + \mathfrak{B}) = 0$ und andererseits $\mathfrak{A} + (\mathfrak{B} \cup \mathfrak{B}^*) = \mathfrak{R}$, d. h. $\delta(\mathfrak{A} + (\mathfrak{B} \cup \mathfrak{B}^*)) = 1$.

In diesem Beispiel ist $m(\mathfrak{B}^*) = 1 > 0$. In einem zweiten soll nun gezeigt werden, daß auch schon die Hinzunahme einer überabzählbaren Menge $\mathfrak{B}^*$ mit $m(\mathfrak{B}^*) = 0$ zu $\mathfrak{B}$ die Dichte von $\mathfrak{A} + \mathfrak{B}$ erhöhen kann: Es sei $\mathfrak{B}$ wieder die Menge der rationalen Punkte und $\mathfrak{B}^*$ die mit der eindimensionalen Menge

$$\mathfrak{B}_1 = \left(x \,\middle|\, x = \sum_{i=-n}^{\infty} a_i 3^{-i};\ a_i = 0, 1;\ n = 1, 2, \ldots\right)$$

gebildete Produktmenge

$$\mathfrak{B}^* = \mathfrak{B}_1 \times \cdots \times \mathfrak{B}_1.$$

Dann ist $m(\mathfrak{B}^*) = 0$ und $\mathfrak{B}^* + \mathfrak{B}^* = \mathfrak{R}$. Nun betrachte man die Summe $\mathfrak{B} + (\mathfrak{B} \cup \mathfrak{B}^*)$. Ihre Dichte wird nicht explizit bestimmt; sie ist entweder > 0 oder $= 0$. Im ersten Fall wählt man $\mathfrak{A} = \mathfrak{B}$ und hat $\delta(\mathfrak{A} + \mathfrak{B}) = \delta(\mathfrak{B} + \mathfrak{B}) = \delta(\mathfrak{B}) = 0$ und

$$\delta(\mathfrak{A} + (\mathfrak{B} \cup \mathfrak{B}^*)) = \delta(\mathfrak{B} + (\mathfrak{B} \cup \mathfrak{B}^*)) > 0.$$

Im zweiten Fall dagegen nimmt man $\mathfrak{A} = \mathfrak{B} \cup \mathfrak{B}^*$ und erhält

$$\delta(\mathfrak{A} + \mathfrak{B}) = \delta((\mathfrak{B} \cup \mathfrak{B}^*) + \mathfrak{B}) = 0$$

sowie

$$\delta(\mathfrak{A} + (\mathfrak{B} \cup \mathfrak{B}^*)) = \delta((\mathfrak{B} \cup \mathfrak{B}^*) + (\mathfrak{B} \cup \mathfrak{B}^*)) \geqq \delta(\mathfrak{B}^* + \mathfrak{B}^*) = \delta(\mathfrak{R}) = 1.$$

§ 2. Basismengen.

6. Jeder Menge $\mathfrak{B} \subset \mathfrak{R}$ werden zwei für alle $\mathfrak{x} \in \mathfrak{R}$ definierte Funktionen $g(\mathfrak{x}) = g_{\mathfrak{B}}(\mathfrak{x})$ und $h(\mathfrak{x}) = h_{\mathfrak{B}}(\mathfrak{x})$ zugeordnet [4]).

Definition 2. 1. *Es sei* $g(\mathfrak{x}) = \min\limits_{n=1,2,\ldots} (n \mid \mathfrak{x} \in n\mathfrak{B})$, *falls es überhaupt ein* n *mit* $\mathfrak{x} \in n\mathfrak{B}$ *gibt. Andernfalls sei* $g(\mathfrak{x}) = \infty$.

Definition 2. 2. $h(\mathfrak{x}) \underset{\mathrm{Df}}{=} \varliminf\limits_{\mathfrak{y} \to \mathfrak{x}} g(\mathfrak{y})$.

Satz 2. 1. $g(\mathfrak{x})$ *und* $h(\mathfrak{x})$ *haben folgende Eigenschaften:*

(2. 1) *Sie sind ganzwertig. Es gilt* $1 \leqq h(\mathfrak{x}) \leqq g(\mathfrak{x}) \leqq \infty$.

(2. 2) *Setzt man*

$$\mathfrak{B}^{(k)} = (\mathfrak{x} \mid g(\mathfrak{x}) \leqq k) \text{ und } \mathfrak{B}^{((k))} = (\mathfrak{x} \mid h(\mathfrak{x}) \leqq k),$$

so ist $\mathfrak{B}^{((k))}$ *abgeschlossen und* $\mathfrak{B}^{((k))} = \widetilde{\mathfrak{B}^{(k)}}$.

(2. 3) $h(\mathfrak{x})$ *ist immer und* $g(\mathfrak{x})$ *sicher für abgeschlossenes* $\mathfrak{B}$ *lebesgue-integrierbar. (Der Integralwert* ∞ *sei dabei zugelassen.)*

Beweis. (2. 1) ist evident. — Nach Definition 2. 2 sind die beiden Aussagen $h(\mathfrak{x}) \leqq k$ und $\exists\, \mathfrak{x}_n \to \mathfrak{x} : g(\mathfrak{x}_n) \leqq k$ äquivalent. Alle $\mathfrak{x}$, die die erste erfüllen, bilden $\mathfrak{B}^{((k))}$; und die, die der zweiten genügen, $\widetilde{\mathfrak{B}^{(k)}}$. Die letzte Menge ist abgeschlossen. Also ergibt sich (2. 2). — Die Integrierbarkeit von $h(\mathfrak{x})$ folgt aus der Abgeschlossenheit, also Meßbarkeit der $\mathfrak{B}^{((k))}$; ebenso die von $g(\mathfrak{x})$ bei abgeschlossenem $\mathfrak{B}$, da dann auch $\mathfrak{B}^{(k)}$ abgeschlossen

[4]) Die folgenden Definitionen übernehme ich mit kleinen Änderungen von Raikov [16] und Chatrovsky [1].

ist. Denn $h(\mathfrak{x})$ und $g(\mathfrak{x})$ sind dann meßbare, nichtnegative Funktionen, und solche sind immer integrierbar (wenn man den Integralwert ∞ zuläßt)[5].

Beispiele zeigen, daß $g(\mathfrak{x})$ im allgemeinen sehr wohl nicht integrierbar sein kann.

Definition 2. 3. *Man nennt die folgenden Zahlen* ($\leqq \infty$)

$$H = \sup_{\mathfrak{x}>0} g(\mathfrak{x}) \quad \textit{(genaue) Ordnung,}$$

$$h = \sup_{\mathfrak{x}>0} h(\mathfrak{x}) \quad \textit{schwache Ordnung,}$$

$$\Lambda = \sup_{\mathfrak{x}>0} \frac{1}{m(\mathfrak{x})} \int_{[0,\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} \quad \textit{(genaue) mittlere Ordnung,}$$

$$\lambda = \sup_{\mathfrak{x}>0} \frac{1}{m(\mathfrak{x})} \int_{[0,\mathfrak{x}]} h(\mathfrak{a})\, d\mathfrak{a} \quad \textit{schwache mittlere Ordnung der Menge } \mathfrak{B}.$$

Dabei sei Λ nur definiert, wenn die Integrale $\int_{[0,\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a}$ alle existieren.

Definition 2. 4. *Man nennt die Zahlen* ($\leqq \infty$)

$$H^* = \inf_{\mathfrak{b}} \sup_{\mathfrak{x}>\mathfrak{b}} g(\mathfrak{x}) \quad \textit{(genaue) asymptotische Ordnung,}$$

$$h^* = \inf_{\mathfrak{b}} \sup_{\mathfrak{x}>\mathfrak{b}} h(\mathfrak{x}) \quad \textit{schwache asymptotische Ordnung,}$$

$$\Lambda^* = \inf_{\mathfrak{b}} \sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} \quad \textit{(genaue) mittlere asymptotische Ordnung,}$$

$$\lambda^* = \inf_{\mathfrak{b}} \sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} h(\mathfrak{a})\, d\mathfrak{a} \quad \textit{schwache mittlere asymptotische Ordnung der Menge } \mathfrak{B}.$$

Hier sei Λ^* nur definiert, wenn $\int_{[\mathfrak{b},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a}$ wenigstens für alle hinreichend großen $\mathfrak{b}$ existiert[6].

7. Unmittelbar aus den Definitionen folgt

Satz 2. 2. *Für die Ordnungen einer Menge $\mathfrak{B}$ gelten die Ungleichungen*

$$(2.4) \qquad 1 \leqq \lambda \leqq \begin{matrix} h \\ \Lambda \end{matrix} \leqq H \leqq \infty; \quad 1 \leqq \lambda^* \leqq \begin{matrix} h^* \\ \Lambda^* \end{matrix} \leqq H^* \leqq \infty;$$

$$(2.5) \qquad H^* \leqq H; \quad h^* \leqq h; \quad \Lambda^* \leqq \Lambda; \quad \lambda^* \leqq \lambda.$$

---

[5]) Vgl. Halmos [4], § 25.

[6]) Die mittlere asymptotische Ordnung wird bei Chatrovsky etwas anders definiert, nämlich durch

$$\overline{\lim_{\mathfrak{b}\to\infty}} \sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} f(\mathfrak{a})\, d\mathfrak{a} \text{ mit } f(\mathfrak{x}) = g(\mathfrak{x}) \text{ oder } = h(\mathfrak{x}).$$

Man sieht aber leicht, daß das mit unserer Definition übereinstimmt: Ist $\mathfrak{b} \leqq \mathfrak{c}$, so folgt

$$\sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} f(\mathfrak{a})\, d\mathfrak{a} \geqq \sup_{\mathfrak{x}>\mathfrak{c}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{c},\mathfrak{x}]} f(\mathfrak{a})\, d\mathfrak{a}$$

und hieraus ergibt sich sofort, daß

$$\lim_{\mathfrak{b}\to\infty} \sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} f(\mathfrak{a})\, d\mathfrak{a}$$

existiert und mit

$$\inf_{\mathfrak{b}} \sup_{\mathfrak{x}>\mathfrak{b}} \frac{1}{m(\mathfrak{x})} \int_{[\mathfrak{b},\mathfrak{x}]} f(\mathfrak{a})\, d\mathfrak{a}$$

übereinstimmt.

Definition 2. 5. *Die Menge $\mathfrak{B}$ heißt bzw. Basis, asymptotische Basis, schwache Basis, schwache asymptotische Basis, wenn bzw. H, H^*, h, h^* endlich ist*[7]).

Dies bedeutet in Formeln bzw.

$$\mathfrak{B}^{(H)} > \mathfrak{M}; \quad \mathfrak{B}^{(H^*)} > [\mathfrak{c}, \infty) \quad \text{für hinreichend großes } \mathfrak{c};$$
$$\mathfrak{B}^{(h)} > \mathfrak{M}; \quad \mathfrak{B}^{(h^*)} > [\mathfrak{c}, \infty)$$

d. h. es lassen sich alle $\mathfrak{x} \in \mathfrak{M}$ mit höchstens H und alle $\mathfrak{x} \in [\mathfrak{c}, \infty)$ sogar mit höchstens H^* Summanden aus $\mathfrak{B}$ darstellen, während die Menge aller $\mathfrak{x}$, die schon mit höchstens h bzw. h^* Summanden darstellbar sind, in $\mathfrak{M}$ bzw. in $[\mathfrak{c}, \infty)$ überall dicht liegt (vgl. (2. 2)).

Satz 2. 3. *Für die einer beliebigen Menge $\mathfrak{B}$ und ihrer abgeschlossenen Hülle $\tilde{\mathfrak{B}}$ zugeordneten Funktionen $h(\mathfrak{x})$, $\tilde{g}(\mathfrak{x})$, $\tilde{h}(\mathfrak{x})$ gilt:*

$$h(\mathfrak{x}) = \tilde{g}(\mathfrak{x}) = \tilde{h}(\mathfrak{x}).$$

Beweis. (a) Aus $\mathfrak{B} \subset \tilde{\mathfrak{B}}$ folgt $\tilde{h}(\mathfrak{x}) \leqq h(\mathfrak{x})$.

(b) Ist $\tilde{g}(\mathfrak{x}) = g < \infty$, so existiert eine Darstellung $\mathfrak{x} = \tilde{\mathfrak{b}}_1 + \cdots + \tilde{\mathfrak{b}}_g$, $\tilde{\mathfrak{b}}_i \in \tilde{\mathfrak{B}}$. In jeder ε-Umgebung von $\tilde{\mathfrak{b}}_i$ gibt es ein $\mathfrak{b}_i \in \mathfrak{B}$, und $\sum_{i=1}^{g} \mathfrak{b}_i$ liegt dann in der $(g\varepsilon)$-Umgebung von $\mathfrak{x}$. Betrachtet man eine Folge $\varepsilon_n \to 0$, so erhält man eine Folge $\mathfrak{x}_n \to \mathfrak{x}$ mit $g(\mathfrak{x}_n) \leqq g$, woraus

$$h(\mathfrak{x}) = \varliminf_{\mathfrak{a} \to \mathfrak{x}} g(\mathfrak{a}) \leqq \varliminf_{\mathfrak{x}_n \to \mathfrak{x}} g(\mathfrak{x}_n) \leqq g = \tilde{g}(\mathfrak{x}),$$

also $h(\mathfrak{x}) \leqq \tilde{g}(\mathfrak{x})$ folgt. Für $\tilde{g}(\mathfrak{x}) = \infty$ ist diese Ungleichung trivial.

(c) Ist $\tilde{h}(\mathfrak{x}) = h < \infty$, so gibt es eine Folge $\mathfrak{x}_n \to \mathfrak{x}$ mit $\tilde{g}(\mathfrak{x}_n) = h$; d. h. $\mathfrak{x}_n = \tilde{\mathfrak{b}}_{n1} + \cdots + \tilde{\mathfrak{b}}_{nh}$, $\tilde{\mathfrak{b}}_{ni} \in \tilde{\mathfrak{B}}$. Die $\tilde{\mathfrak{b}}_{ni}$ sind beschränkt, so daß eine Teilfolge n' existiert, für die die $\tilde{\mathfrak{b}}_{n'i} \to \tilde{\mathfrak{b}}_i$ konvergieren. Dann ist $\mathfrak{x} = \tilde{\mathfrak{b}}_1 + \cdots + \tilde{\mathfrak{b}}_h$ und $\tilde{\mathfrak{b}}_i \in \tilde{\mathfrak{B}}$ wegen der Abgeschlossenheit von $\tilde{\mathfrak{B}}$; also ist $\tilde{g}(\mathfrak{x}) \leqq h = \tilde{h}(\mathfrak{x})$. Auch diese Ungleichung ist für $\tilde{h}(\mathfrak{x}) = \infty$ trivial und daher allgemeingültig.

Zusammen folgt aus (a) bis (c)

$$\tilde{h}(\mathfrak{x}) \leqq h(\mathfrak{x}) \leqq \tilde{g}(\mathfrak{x}) \leqq \tilde{h}(\mathfrak{x}),$$

also die Behauptung.

8. Ist $\mathfrak{A}$ eine abgeschlossene Menge und $\mathfrak{B}$ eine überall dichte Teilmenge von $\mathfrak{A}$, für die also $\tilde{\mathfrak{B}} = \mathfrak{A}$ gilt, so wird nach Satz 2. 3

$$h_{\mathfrak{B}}(\mathfrak{x}) = g_{\mathfrak{A}}(\mathfrak{x}) = h_{\mathfrak{A}}(\mathfrak{x}).$$

Folglich stimmen die schwachen Ordnungen von $\mathfrak{B}$ mit den entsprechenden schwachen und genauen Ordnungen von $\mathfrak{A}$ überein. Insbesondere ist $\mathfrak{B}$ genau dann schwache (asymptotische) Basis gegebener schwacher Ordnungen, wenn $\mathfrak{A}$ gewöhnliche (asymptotische) Basis derselben genauen Ordnungen ist. Also gilt

---

[7]) Wir nennen eine Menge also nur dann (schwache, asymptotische) Basis, wenn die entsprechende Ordnung endlich ist. Raikow betrachtete auch (schwache, asymptotische) Basen unendlicher Ordnung, d. h. Mengen, mit denen sich ebenfalls die Punkte $\mathfrak{x}$ der oben genannten Bereiche als endliche Summen darstellen lassen, wobei aber die benötigte Summandenzahl nicht mehr beschränkt ist. Für solche Mengen werden jedoch, da ihre Ordnungen und dann nach Satz 2. 5, Folgerung 1 auch ihre mittleren Ordnungen unendlich sind, alle im folgenden bewiesenen Ungleichungen, die Basisordnungen enthalten, trivial.

Satz 2. 4. *Die Klasse aller (asymptotischen) schwachen Basen der schwachen (asymptotischen) Ordnungen h, λ (bzw. h^*, λ^*) stimmt mit der Klasse aller überall dichten Teilmengen von abgeschlossenen (asymptotischen) Basen der genauen (asymptotischen) Ordnungen $H = h$, $\Lambda = \lambda$ (bzw. $H^* = h^*$, $\Lambda^* = \lambda^*$) überein*[8]).

Damit ist die Untersuchung der Klasse aller schwachen Basen auf die der abgeschlossenen gewöhnlichen Basen zurückgeführt; ebenso die Untersuchung der schwachen Ordnungen einer beliebigen Menge $\mathfrak{B}$ auf die der genauen Ordnungen von $\widetilde{\mathfrak{B}}$.

Satz 2. 5. *Ist Λ bzw. Λ^* für $\mathfrak{B}$ definiert, so gilt*

$$H \leqq 2\Lambda; \quad H^* \leqq 2\Lambda^*. \tag{2. 6}$$

Ist darüber hinaus H bzw. H^* endlich, so ist

$$\begin{aligned} \Lambda &= H - \sigma\,(\mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(H-1)}), \\ \Lambda^* &= H^* - \sigma^*(\mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(H^*-1)}). \end{aligned} \tag{2. 7}$$

Beweis[9]). Zu (2. 6): Die ersichtlich richtige Ungleichung $g(\mathfrak{a} + \mathfrak{b}) \leqq g(\mathfrak{a}) + g(\mathfrak{b})$ ergibt mit $\mathfrak{a} + \mathfrak{b} = \mathfrak{x}$ bei Integration über $[\mathfrak{o}, \mathfrak{x}]$

$$m(\mathfrak{x})\, g(\mathfrak{x}) \leqq \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} + \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{x} - \mathfrak{a})\, d\mathfrak{a} = 2 \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a},$$

also

$$H = \sup_{\mathfrak{x} > \mathfrak{o}} g(\mathfrak{x}) \leqq 2 \sup_{\mathfrak{x} > \mathfrak{o}} \frac{1}{m(\mathfrak{x})} \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} = 2\Lambda.$$

Zu (2. 7): Daß Λ definiert ist, bedeutet nach Definition 2. 3, daß alle Integrale $\int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a}$ existieren. Ist nun $\mathfrak{B}_k \underset{\mathrm{Df}}{=} (\mathfrak{x} \mid g(\mathfrak{x}) = k)$, so sind die $\mathfrak{B}_k$ elementefremd und nach der Theorie des lebesgueschen Integrals[10]) alle Mengen $\mathfrak{B}_k \cap [\mathfrak{o}, \mathfrak{x}]$ meßbar. Daher gilt

$$\begin{aligned} \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} &= \sum_{k=1}^{H} k\, m(\mathfrak{B}_k, \mathfrak{x}) = \sum_{j=1}^{H} \sum_{k=j}^{H} m(\mathfrak{B}_k, \mathfrak{x}) \\ &= \sum_{j=1}^{H} m(\mathfrak{B}_j \cup \cdots \cup \mathfrak{B}_H, \mathfrak{x}) = H\, m(\mathfrak{x}) - \sum_{j=2}^{H} m(\mathfrak{B}^{(j-1)}, \mathfrak{x}), \end{aligned}$$

$$\Lambda = \sup_{\mathfrak{x} > \mathfrak{o}} \frac{1}{m(\mathfrak{x})} \int\limits_{[\mathfrak{o},\mathfrak{x}]} g(\mathfrak{a})\, d\mathfrak{a} = H - \inf_{\mathfrak{x} > \mathfrak{o}} \frac{\sum\limits_{i=1}^{H-1} m(\mathfrak{B}^{(i)}, \mathfrak{x})}{m(\mathfrak{x})} = H - \sigma(\mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(H-1)}).$$

---

[8]) Für Basen unendlicher Ordnung (vgl. Fußnote 7) ist der entsprechende Zusammenhang nicht mehr gültig. (Asymptotische) Basen unendlicher Ordnung sind nämlich nach Definition durch H (bzw. H^*) $= \infty$ und „$g(\mathfrak{x}) < \infty$ für alle (hinreichend großen) $\mathfrak{x}$" charakterisiert, während für schwache (asymptotische) Basen unendlicher Ordnung neben h (bzw. h^*) $= \infty$ noch „$g(\mathfrak{x}) < \infty$ für alle $\mathfrak{x}$ einer überall dichten Teilmenge von $\mathfrak{R}$ (bzw. von $[\mathfrak{c}, \infty)$ mit genügend großem $\mathfrak{c}$)" kennzeichend ist. Dafür wieder ist „$h(\mathfrak{x}) < \infty$ für alle (genügend großen) $\mathfrak{x}$" zwar hinreichend, aber nicht notwendig, so daß die Klasse aller überall dichten Teilmengen von abgeschlossenen (asymptotischen) Basen unendlicher Ordnung nur eine echte Teilklasse der Klasse aller schwachen (asymptotischen) Basen unendlicher Ordnung ist.

[9]) Die Grundidee des Beweises von (2. 6) stammt von I. Vinogradow (vgl. Raikow [16]). Eine Beweisskizze für unsern Fall findet sich bei Chatrovsky [1]. Der Beweis von (2. 7) ist den entsprechenden Überlegungen in der gewöhnlichen additiven Zahlentheorie nachgebildet (vgl. A. Stöhr [21], H. Klöter u. A. Stöhr [9]).

[10]) Vgl. Halmos [4], Kapitel IV, V.

Die asymptotischen Formeln beweist man entsprechend. Zu (2. 7) hat man beim letzten Schritt die für alle von $\mathfrak{b}$ unabhängigen Funktionen $f(\mathfrak{x})$ gültige Beziehung

$$\sup_{\mathfrak{b}} \inf_{\mathfrak{x}>\mathfrak{b}} f(\mathfrak{x}) = \varliminf_{\mathfrak{x}\to\infty} f(\mathfrak{x})$$

zu verwenden.

9. Folgerungen. *1.* Aus (2. 4) und (2. 6) ergibt sich, daß Λ bzw. Λ^* dann und nur dann endlich ist, wenn H bzw. H^* es ist, d. h. wenn $\mathfrak{B}$ (asymptotische) Basis ist.

2. Bei endlichem H, H^* ist wegen (2. 7) genau dann $H = \Lambda$ bzw. $H^* = \Lambda^*$, wenn $\delta(\mathfrak{B}^{(H-1)}) = 0$ bzw. $\delta^*(\mathfrak{B}^{(H^*-1)}) = 0$ ist. Denn es gilt

$$\delta(\mathfrak{B}^{(H-1)}) \leqq \sigma(\mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(H-1)}) \leqq (H-1)\,\delta(\mathfrak{B}^{(H-1)})$$

und Analoges für die asymptotischen Dichten.

3. Mit $\beta = \delta(\mathfrak{B})$ gelten bei endlichem H, H^* die Abschätzungen

$$2-\beta \leqq \Lambda \leqq H-(H-1)\,\beta, \tag{2. 8}$$

$$2-\beta^* \leqq \Lambda^* \leqq H^*-(H^*-1)\,\beta^*. \tag{2. 9}$$

Denn es gilt

$$\begin{aligned}(H-1)\,\beta = \sigma(\mathfrak{B}, \ldots, \mathfrak{B}) &\leqq \sigma(\mathfrak{B}^{(1)}, \ldots, \mathfrak{B}^{(H-1)})\\ &\leqq \sigma(\mathfrak{B}, \mathfrak{N}, \ldots, \mathfrak{N}) = \beta + H - 2\end{aligned}$$

und Analoges für die asymptotischen Dichten.

Wendet man Satz 2. 5 auf $\tilde{\mathfrak{B}}$ an und benutzt die Sätze 2. 3, 2. 4, so erhält man für die schwachen Ordnungen von $\mathfrak{B}$ folgende Aussagen:

Satz 2. 6. *Ist $\mathfrak{B}$ eine beliebige Menge und $\tilde{\beta} = \delta(\tilde{\mathfrak{B}})$, $\tilde{\beta}^* = \delta^*(\tilde{\mathfrak{B}})$, so gilt*

$$h \leqq 2\lambda, \quad h^* \leqq 2\lambda^*; \tag{2. 10}$$

(2. 11) *λ, λ^* sind dann und nur dann endlich, wenn h, h^* es sind, d. h. wenn $\mathfrak{B}$ schwache (asymptotische) Basis ist.*

Sind h, h^* endlich, so ist weiter

$$\begin{aligned}\lambda &= h - \sigma(\mathfrak{B}^{((1))}, \ldots, \mathfrak{B}^{((h-1))}),\\ \lambda^* &= h^* - \sigma^*(\mathfrak{B}^{((1))}, \ldots, \mathfrak{B}^{((h^*-1))}),\end{aligned} \tag{2. 12}$$

(2. 13) *$h = \lambda$ bzw. $h^* = \lambda^*$ genau dann, wenn $\delta(\mathfrak{B}^{((h-1))}) = 0$ bzw. $\delta^*(\mathfrak{B}^{((h-1))}) = 0$ ist,*

$$\begin{aligned}2-\tilde{\beta} &\leqq \lambda \leqq h-(h-1)\,\tilde{\beta},\\ 2-\tilde{\beta}^* &\leqq \lambda^* \leqq h^*-(h^*-1)\,\tilde{\beta}^*.\end{aligned} \tag{2. 14}$$

§ 3. Basiskriterien.

10. Satz 3. 1. *Hat eine Menge $\mathfrak{B}$ positive Dichte $\beta > 0$, so ist $\mathfrak{B}$ finite Basis; hat ihre abgeschlossene Hülle $\tilde{\mathfrak{B}}$ positive Dichte $\tilde{\beta} > 0$, so ist $\mathfrak{B}$ schwache finite Basis. Haben $\mathfrak{B}$ bzw. $\tilde{\mathfrak{B}}$ asymptotische Dichten $\beta^*, \tilde{\beta}^* > \frac{1}{2}$, so ist $\mathfrak{B}$ asymptotische (schwache) Basis. Für die Ordnungen gelten die Abschätzungen:*

(3. 1) *Ist bzw. $\mathfrak{B} > \mathfrak{M}$, $\tilde{\mathfrak{B}} > \mathfrak{M}$, $\mathfrak{B} > [\mathfrak{c}, \infty)$, $\tilde{\mathfrak{B}} > [\mathfrak{c}, \infty)$ für ein passendes $\mathfrak{c}$, so gilt bzw.*

$$H = \Lambda = 1, \quad h = \lambda = 1, \quad H^* = \Lambda^* = 1, \quad h^* = \lambda^* = 1.$$

(3. 2) *Treffen die entsprechenden Voraussetzungen von* (3. 1) *nicht zu, ist aber* $\beta, \tilde{\beta}, \beta^*, \tilde{\beta}^* > \frac{1}{2}$, *so gilt*

$$H, h, H^*, h^* = 2,$$

$$\Lambda = 2 - \beta, \quad \lambda = 2 - \beta, \quad \Lambda^* = 2 - \beta^*, \quad \lambda^* = 2 - \tilde{\beta}^*.$$

(3. 3) *Ist* $0 < \beta \leqq \frac{1}{2}$, *so gilt für abgeschlossenes* $\mathfrak{B}$

$$\text{(A)} \qquad H \leqq r \left\langle \frac{1}{\beta} \right\rangle^{11)}, \quad \Lambda \leqq \frac{r - (r-1)\beta^2}{\beta}$$

und für beliebiges $\mathfrak{B}$

$$\text{(B)} \qquad \Lambda \leqq H \leqq 5 \frac{r}{F(r)} \frac{1}{\beta} \ln \frac{1}{\beta}.$$

Dabei ist $F(r)$ *die Funktion aus Satz* 4. 2.

Ferner ist für beliebiges $\mathfrak{B}$ *mit* $0 < \tilde{\beta} \leqq \frac{1}{2}$

$$\text{(C)} \qquad h \leqq r \left\langle \frac{1}{\tilde{\beta}} \right\rangle, \quad \lambda \leqq \frac{r - (r-1)\beta^2}{\tilde{\beta}}.$$

Der Beweisgang ist folgender: (3. 1) ist trivial. (3. 2) folgt aus Satz 3. 2, der mit einer in der additiven Zahlentheorie geläufigen Schlußweise [12]) bewiesen wird. Zu (3. 3) zeige ich zunächst den Hilfssatz 3. 1, der für abgeschlossene $\mathfrak{B}$ die Zurückführung des r-dimensionalen Falles auf den eindimensionalen erlaubt. Das ergibt (A). Mit Satz 2. 4 folgt hieraus (C). (B) erhält man schließlich, wenn man (C) mit der Formel (4. 3) aus dem II. Kapitel kombiniert.

Bemerkung 1. Die Abschätzungen (3. 3) dürften recht grob sein. Das Hauptgewicht liegt daher auf der Feststellung, daß überhaupt Abschätzungen der Basisordnungen durch die Dichten existieren.

Bemerkung 2. Man wird vermuten, daß zu (3. 3) analoge Aussagen auch für den asymptotischen Fall möglich sind. Mir ist hierzu jedoch kein Beweis bekannt. Eine Hilfssatz 3. 1 entsprechende Zurückführung auf den eindimensionalen Fall dürfte kaum durchführbar sein.

11. Satz 3. 2. *Für beliebige Mengen* $\mathfrak{A}, \mathfrak{B}$ *mit* $\sigma(\mathfrak{A}, \mathfrak{B}) > 1$ *bzw.* $\sigma^*(\mathfrak{A}, \mathfrak{B}) > 1$ *folgt*

$$\mathfrak{A} + \mathfrak{B} > \mathfrak{M} \text{ bzw. } \mathfrak{A} + \mathfrak{B} > [\mathfrak{c}, \infty) \text{ für hinreichend großes } \mathfrak{c}.$$

Beweis. Nach Hilfssatz 1. 1 genügt es, meßbare $\mathfrak{A}, \mathfrak{B}$ zu betrachten. Ist nun $\mathfrak{x} \notin \mathfrak{A} + \mathfrak{B}$ und $\mathfrak{a} \in \mathfrak{A}$, $\mathfrak{a} \leqq \mathfrak{x}$, so muß $\mathfrak{x} - \mathfrak{a} \notin \mathfrak{B}$ sein. Also gilt $\chi(\mathfrak{A}, \mathfrak{a}) \leqq \chi(\bar{\mathfrak{B}}, \mathfrak{x} - \mathfrak{a})$ und hieraus folgt

$$\chi(\mathfrak{A}, \mathfrak{a}) + \chi(\mathfrak{B}, \mathfrak{x} - \mathfrak{a}) \leqq 1,$$

$$\text{(3. 4)} \qquad \int_{[\mathfrak{o}, \mathfrak{x}]} \chi(\mathfrak{A}, \mathfrak{a})\, d\mathfrak{a} + \int_{[\mathfrak{o}, \mathfrak{x}]} \chi(\mathfrak{B}, \mathfrak{x} - \mathfrak{a})\, d\mathfrak{a} = m(\mathfrak{A}, \mathfrak{x}) + m(\mathfrak{B}, \mathfrak{x}) \leqq m(\mathfrak{x}).$$

Wäre nun $\mathfrak{A} + \mathfrak{B} \not> \mathfrak{M}$, so gäbe es ein $\mathfrak{x}_0 > \mathfrak{o}$, $\mathfrak{x}_0 \notin \mathfrak{A} + \mathfrak{B}$ und aus (3. 4) folgte

$$1 < \sigma(\mathfrak{A}, \mathfrak{B}) \leqq \frac{m(\mathfrak{A}, \mathfrak{x}_0) + m(\mathfrak{B}, \mathfrak{x}_0)}{m(\mathfrak{x}_0)} \leqq 1.$$

---

[11]) $\langle a \rangle$ bedeutet die kleinste ganze Zahl n mit $a \leqq n$.

[12]) Vgl. z. B. Ostmann [15], Kapitel 9, Satz 3.

Gäbe es ferner kein $\mathfrak{c}$, derart daß $\mathfrak{A} + \mathfrak{B} > [\mathfrak{c}, \infty)$ ist, so könnte man eine Folge $\mathfrak{x}_n \to \infty$ mit $\mathfrak{x}_n \notin \mathfrak{A} + \mathfrak{B}$ finden und erhielte

$$1 < \sigma^*(\mathfrak{A}, \mathfrak{B}) \leqq \varliminf_{\mathfrak{x}_n \to \infty} \frac{m(\mathfrak{A}, \mathfrak{x}_n) + m(\mathfrak{B}, \mathfrak{x}_n)}{m(\mathfrak{x}_n)} \leqq 1.$$

Das wäre aber beide Male ein Widerspruch.

Setzt man nun $\mathfrak{A} = \mathfrak{B}$ und ist $\beta, \beta^* > \frac{1}{2}$, so ergibt sich wegen $\sigma(\mathfrak{B}, \mathfrak{B}) = 2\beta > 1$; $\sigma^*(\mathfrak{B}, \mathfrak{B}) = 2\beta^* > 1$ aus Satz 3. 2, daß $2\mathfrak{B} > \mathfrak{M}$ bzw. $> [\mathfrak{c}, \infty)$, also $H, H^* \leqq 2$ ist. Da die Fälle mit $H, H^* = 1$ offenbar unter (3. 1) erledigt sind, gilt in (3. 2) $H, H^* = 2$. Die Aussagen über Λ, Λ^* folgen dann aus (2. 8), (2. 9) und die über die schwachen Ordnungen $h, h^*, \lambda, \lambda^*$ aus Satz 2. 4 durch Anwendung auf $\tilde{\mathfrak{B}}$.

12. Hilfssatz 3. 1. *Ist $\mathfrak{B}$ eine abgeschlossene Menge mit der finiten Dichte β und $\mathfrak{B}'$ die Menge aller Punkte von $\mathfrak{B}$ auf einer der Koordinatenachsen, so gilt für die Dichte β' von $\mathfrak{B}'$ (als eindimensionale Menge betrachtet): $\beta' \geqq \beta$.*

Beweis. $\mathfrak{B}$ und $\mathfrak{B}'$ sind abgeschlossen, also meßbar. m bezeichne das r-dimensionale und m_1 das eindimensionale Maß. Ich betrachte z. B. die x_1-Achse.

Es sei im Gegensatz zur Behauptung $\beta' < \beta$. Dann gibt es ein $x > 0$ und ein $\varepsilon > 0$, so daß $m_1(\mathfrak{B}', x) \leqq (\beta - 3\varepsilon)\, x$ gilt. Außerdem existiert sicher eine Intervallüberdeckung

$$\mathfrak{U}' = \bigcup_{i=1}^{\infty} \mathfrak{U}_i', \quad \mathfrak{U}_i' = [a_i, b_i)$$

zu $\mathfrak{B}'$ mit $m_1(\mathfrak{U}', x) \leqq m_1(\mathfrak{B}', x) + \varepsilon x$. Zu den $\mathfrak{U}_i'$ werden umfassendere Intervalle

$$\mathfrak{B}_i' = \left[a_i - \frac{\varepsilon x}{2^{i+1}}, \; b_i + \frac{\varepsilon x}{2^{i+1}}\right)$$

definiert. Für $\mathfrak{B}' = \bigcup_{i=1}^{\infty} \mathfrak{B}_i'$ gilt dann

$$m_1(\mathfrak{B}', x) \leqq m_1(\mathfrak{U}', x) + \varepsilon x \leqq (\beta - \varepsilon)\, x, \tag{3. 5}$$

und alle Punkte von $\mathfrak{B}'$ sind innere Punkte von $\mathfrak{B}'$.

Definiert man nun eine r-dimensionale Menge $\mathfrak{B}$ durch

$$\mathfrak{B} = (\mathfrak{x} \mid \mathfrak{x} = (v_1, x_2, \ldots, x_r);\; v_1 \in \mathfrak{B}';\; 0 \leqq x_i < \infty,\; i = 2, \ldots, r)$$

und setzt $\mathfrak{x}_\delta = (x, \delta, \ldots, \delta)$ mit $\delta > 0$, so erhält man

$$\begin{aligned} m(\mathfrak{B}, \mathfrak{x}_\delta) = \delta^{r-1} m_1(\mathfrak{B}', x) &\leqq \delta^{r-1}(\beta - \varepsilon)\, x \\ &= (\beta - \varepsilon)\, m(\mathfrak{x}_\delta) < m(\mathfrak{B}, \mathfrak{x}_\delta). \end{aligned}$$

Folglich ist $m(\mathfrak{B} - \mathfrak{B}, \mathfrak{x}_\delta) > 0$, d. h. $(\mathfrak{B} - \mathfrak{B}) \cap [\mathfrak{o}, \mathfrak{x}_\delta]$ ist für alle $\delta > 0$ nicht leer, d. h. $\mathfrak{B} - \mathfrak{B}$ hat Häufungspunkte auf der x_1-Achse. Ein solcher Häufungspunkt $\mathfrak{b}$ muß wegen der Abgeschlossenheit zu $\mathfrak{B}$ und $\mathfrak{B}'$ gehören und ist also nach Konstruktion von $\mathfrak{B}'$ innerer Punkt von $\mathfrak{B}'$. Also gibt es eine r-dimensionale Umgebung von $\mathfrak{b}$, deren Durchschnitt mit $\mathfrak{N}$ ganz in $\mathfrak{B}$ liegt. $\mathfrak{b}$ kann somit gar nicht Häufungspunkt von $\mathfrak{B} - \mathfrak{B}$ sein. Widerspruch.

Der Hilfssatz 3. 1 erlaubt nunmehr, die nur für den eindimensionalen Fall bewiesene Übertragung des Mannschen Satzes für unseren Fall, d. h. auf $\mathfrak{B}'$ anzuwenden. Die Vor-

aussetzung $0 \in \mathfrak{B}'$ ist erfüllt, denn wegen $\delta(\mathfrak{B}) = \beta > 0$ und $\mathfrak{B} = \tilde{\mathfrak{B}}$ ist $0 \in \mathfrak{B}$. Man erhält

$$\delta\left(\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}'\right) \geqq \min\left(1, \left\langle\frac{1}{\beta}\right\rangle \delta(\mathfrak{B}')\right) = \min\left(1, \left\langle\frac{1}{\beta}\right\rangle \beta'\right) = 1,$$

d. h.

$$m_1\left(\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}', x\right) = x \text{ für alle } x, \text{ d. h. } m_1\left(\overline{\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}'}\right) = 0.$$

Hieraus folgt wegen der Abgeschlossenheit von $\mathfrak{B}'$, die die Abgeschlossenheit von $\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}'$ und die Offenheit von $\overline{\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}'}$ nach sich zieht, $\overline{\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}'} = \emptyset$. Da diese Betrachtung für jede der r Koordinatenachsen gilt, enthält $\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B}$ sämtliche Koordinatenachsen vollständig. Die Zerlegung

$$\mathfrak{x} = (x_1, 0, \ldots, 0) + \cdots + (0, \ldots, 0, x_r)$$

ergibt schließlich

$$\mathfrak{N} \subset \sum_{i=1}^{r} \left\langle\frac{1}{\beta}\right\rangle \mathfrak{B} = r\left\langle\frac{1}{\beta}\right\rangle \mathfrak{B},$$

also

$$H \leqq r\left\langle\frac{1}{\beta}\right\rangle.$$

Die Ungleichung für Λ folgt dann aus (2. 8). Damit ist (A) bewiesen.

Wendet man (A) auf die abgeschlossene Menge $\tilde{\mathfrak{B}}$ an, so ergibt sich mit Satz 2. 4 sofort (C).

13. Um endlich (B) zu zeigen, benutze ich Satz 4. 2, Formel (4. 3) aus dem nächsten Kapitel: Für alle $\mathfrak{A}, \mathfrak{B}$ gilt

$$\delta(\mathfrak{A} + \mathfrak{B}) \geqq \alpha\left(1 + F(r)\frac{1-\alpha}{\lambda}\right). \tag{3. 6}$$

Setzt man $\delta(\varkappa\mathfrak{B}) = \beta_\varkappa$ und bestimmt das kleinste ganzzahlige $\varkappa_0$ mit

$$\beta\left(1 + \frac{F(r)}{2\lambda}\right)^{\varkappa_0} > \frac{1}{2},$$

so gibt es ein $\varkappa_1 \leqq \varkappa_0 + 1$ mit $\beta_{\varkappa_1} > \frac{1}{2}$. Denn entweder gibt es bereits ein solches $\varkappa_1 \leqq \varkappa_0$, oder für alle $\varkappa \leqq \varkappa_0$ ist $\beta_\varkappa \leqq \frac{1}{2}$. Für diese erhält man dann aus (3. 6) mit $\mathfrak{A} = \varkappa\mathfrak{B}$

$$\beta_{\varkappa+1} \geqq \beta_\varkappa\left(1 + F(r)\frac{1-\beta_\varkappa}{\lambda}\right) \geqq \beta_\varkappa\left(1 + \frac{F(r)}{2\lambda}\right),$$

also

$$\beta_{\varkappa_0+1} \geqq \beta\left(1 + \frac{F(r)}{2\lambda}\right)^{\varkappa_0} > \frac{1}{2}.$$

Nach Satz 3. 2 ist also $H \leqq 2\varkappa_1 \leqq 2\varkappa_0 + 2$.

Die Bedingung für $\varkappa_0$ bedeutet

$$\frac{\ln\frac{1}{2\beta}}{\ln\left(1+\frac{F(r)}{2\lambda}\right)} < \varkappa_0 \leqq \frac{\ln\frac{1}{2\beta}}{\ln\left(1+\frac{F(r)}{2\lambda}\right)} + 1.$$

Somit gilt

$$H \leqq \frac{2\ln\frac{1}{2\beta}}{\ln\left(1+\frac{F(r)}{2\lambda}\right)} + 4 \leqq 2\ln\frac{1}{2\beta}\left(\frac{2\lambda}{F(r)}+1\right)+4$$

und mit $\lambda \geqq 1$ und $F(r) \leqq \frac{1}{2}$ (nach (4. 5), (4. 6))[13]

$$H \leqq 5\,\frac{\lambda}{F(r)}\ln\frac{1}{2\beta} + 4 \leqq 5\,\frac{\lambda}{F(r)}\ln\frac{1}{\beta}$$

und wegen $\lambda \leqq \frac{r-(r-1)\tilde{\beta}^2}{\tilde{\beta}} \leqq \frac{r}{\beta}$ (nach (3. 3, C)) schließlich

$$H \leqq 5\,\frac{r}{F(r)}\,\frac{1}{\tilde{\beta}}\ln\frac{1}{\beta}.$$

II. Kapitel: Summen einer Basis und einer beliebigen Menge.

§ 4. Die Hauptformel.

14. Wie schon in der Einleitung erwähnt, bewies Chatrovsky [1] folgenden Satz:

Satz von Chatrovsky. *Ist $\mathfrak{A}$ eine beliebige Menge mit den Dichten α, α^* und $\mathfrak{B}$ eine schwache (asymptotische) Basis mit der mittleren (asymptotischen) Ordnung λ, λ^*, für die $\mathfrak{o} \in \mathfrak{B}$ gilt, so gelten für die Dichten der Summenmenge $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$ die Abschätzungen*

$$\gamma \geqq \alpha\left(1+\frac{1-2^{r+1}\alpha}{2^r\lambda}\right), \tag{4. 1}$$

$$\gamma^* \geqq \alpha^*\left(1+\frac{1-2^{r-1}\alpha^*}{2^r\lambda^*}\right). \tag{4. 2}$$

Dieses Ergebnis ist nur für $0 < \alpha, \alpha^* < 2^{r-1}$ nichttrivial. Ziel dieses Kapitels ist es, folgende ähnliche Ungleichungen herzuleiten, die für alle $0 < \alpha, \alpha^* < 1$ sinnvoll sind.

Satz 4. 1. *Ist $\mathfrak{A}$ eine beliebige Menge (Dichten α, α^*) und $\mathfrak{B}$ eine (asymptotische) schwache Basis (mittlere Ordnungen λ, λ^*), für die im asymptotischen Fall noch $\mathfrak{o} \in \tilde{\mathfrak{B}}$ vorausgesetzt sei, und $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$, so gilt*

$$\gamma \geqq \alpha\left(1+F(r)\,\frac{1-\alpha}{\lambda}\right), \tag{4. 3}$$

$$\gamma^* \geqq \alpha^*\left(1+F(r)\,\frac{1-\alpha^*}{\lambda^*}\right). \tag{4. 4}$$

---

[13]) Denn es ist $x_0 > 0$ und daher

$$F(r) = 2^r x_0^r\left(1-\sum_u\binom{r}{u}x_0^u\right) \leqq 2^r x_0^r(1-rx_0) \leqq \max_{x>0}(2^r x^r(1-rx)) = \frac{2^r}{(r+1)^{r+1}}.$$

Diese Abschätzung für $F(r)$ ist übrigens für $1 \leqq r \leqq 6$ besser als die in (4. 7) angegebene.

Dabei ist

$$F(r) = 2^r x_0^r \left(1 - \sum_{u \text{ ungerade}} \binom{r}{u} x_0^u\right), \tag{4.5}$$

wo x_0 die eindeutig bestimmte reelle Lösung von

$$\sum_{u \text{ ungerade}} \binom{r}{u} (r+u)\, x_0^u = r \tag{4.6}$$

ist.

Satz 4. 2. *Mit*

$$w_0 = \ln\left(1 + \sqrt{2}\right) = 0{,}8814\ldots,$$

$$v_0 = \frac{1}{2\sqrt{2}}(2 - w_0) = 0{,}3955\ldots,$$

$$c = \frac{1}{\sqrt{2}} e^{\frac{w_0}{2\sqrt{2}} - 1} = 0{,}3552\ldots$$

gelten für die Funktion $F(r)$ in Satz 4. 1 die Abschätzungen

$$\frac{1}{2}\left(\frac{2w_0}{r + w_0}\right)^{r+1} \leqq F(r) \leqq \frac{1}{\sqrt{2}}\left(\frac{2w_0}{r + v_0}\right)^{r+1} \tag{4.7}$$

und die asymptotische Aussage

$$F(r) = c\left(\frac{2w_0}{r}\right)^{r+1}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{4.8}$$

Bemerkung. Insbesondere gilt $F(1) = \frac{1}{2}$, so daß die Ungleichungen (4. 3), (4. 4) für den eindimensionalen Fall mit den Formeln (4. 1), (4. 2) von Chatrovsky zusammenfallen. Für $r \geqq 2$ sind sie dagegen außer für sehr kleine α, α^* besser als Chatrovskys Ergebnis.

Unter der zusätzlichen Voraussetzung, daß die natürliche Dichte α^* von $\mathfrak{A}$ existiert, gilt noch folgender Satz, der für alle $0 < \alpha^* < 1$ besser als (4. 2) und (4. 4) ist:

Satz 4. 3. *Ist $\mathfrak{A}$ eine Menge, deren natürliche Dichte α^* existiert, und ist $\mathfrak{B}$ eine asymptotische schwache Basis mit der mittleren Ordnung λ^* und $\mathfrak{o} \in \widetilde{\mathfrak{B}}$, so gilt für die asymptotische Dichte der Summenmenge $\mathfrak{C} = \mathfrak{A} + \mathfrak{B}$*

$$\gamma^* \geqq \alpha^*\left(1 + \frac{1 - \alpha^*}{2^r \lambda^*}\right). \tag{4.9}$$

15. Der Nachweis dieser Sätze erfordert längere Betrachtungen. Die Sätze 4. 1, 4. 3 werden in den §§ 4, 5 und Satz 4. 2 in § 6 bewiesen.

Wegen Hilfssatz 1. 1 kann $\mathfrak{A}$ als meßbar und wegen Hilfssatz 1. 2 noch $\mathfrak{o} \in \mathfrak{B}$ angenommen werden; letzteres, da $\mathfrak{o} \in \widetilde{\mathfrak{B}}$ im asymptotischen Fall in den Sätzen 4. 1, 4. 3 vorausgesetzt wird und im finiten Fall immer erfüllt ist. Die charakteristische Funktion $\chi(\mathfrak{A}, \mathfrak{a})$ von $\mathfrak{A}$ ist dann im lebesgueschen Sinne integrierbar. Man definiert für alle $\mathfrak{x} \in \mathfrak{R}$ und alle $\mathfrak{o} \leqq \mathfrak{m} \leqq \mathfrak{x}$ die Funktion

$$D(\mathfrak{m}) = D_{\mathfrak{x}}(\mathfrak{m}) = \int_{\mathfrak{D}[\mathfrak{o}, \mathfrak{x} - \mathfrak{m}]} \chi(\mathfrak{A}, \mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a} + \mathfrak{m})\, d\mathfrak{a} \tag{4.10}$$

Hilfssatz 4. 1. *$D_{\mathfrak{x}}(\mathfrak{m})$ ist eine stetige Funktion von $\mathfrak{m}$.*

Beweis. $\mathfrak{t} \in \mathfrak{R}$ sei ein beliebiger Punkt, für den $D(\mathfrak{m}+\mathfrak{t})$ definiert ist. Dann gilt

$$
\begin{aligned}
&|D(\mathfrak{m}+\mathfrak{t}) - D(\mathfrak{m})| \\
&\quad \leqq \Big| \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{m}+\mathfrak{t})\, d\mathfrak{a} - \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{m})\, d\mathfrak{a} \Big| \\
&\qquad + \Big| \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\, \overline{\chi}(\mathfrak{A},\mathfrak{a}+\mathfrak{m})\, d\mathfrak{a} - \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}]} \chi(\mathfrak{A},\mathfrak{a})\, \overline{\chi}(\mathfrak{A},\mathfrak{a}+\mathfrak{m})\, d\mathfrak{a} \Big| \\
&\quad \leqq \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}-\mathfrak{t}]} 1 \,|\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{m}+\mathfrak{t}) - \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{m}) \,|\, d\mathfrak{a} \\
&\qquad + \Big| \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{m}-\mathfrak{t}]\,\Delta\,[\mathfrak{o},\mathfrak{x}-\mathfrak{m}]} \chi(\mathfrak{A},\mathfrak{a})\, \overline{\chi}(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{m})\, d\mathfrak{a} \Big| .
\end{aligned}
$$

Diese Ausdrücke gehen für $\mathfrak{t} \to \mathfrak{o}$ selbst $\to 0$.

Hilfssatz 4. 2. *Für alle* $\mathfrak{s}, \mathfrak{t} \in \mathfrak{R}$ *mit* $\mathfrak{o} \leqq \mathfrak{s}+\mathfrak{t} \leqq \mathfrak{x}$ *gilt*

$$D(\mathfrak{s}+\mathfrak{t}) \leqq D(\mathfrak{s}) + D(\mathfrak{t}). \tag{4. 11}$$

Beweis.

$$
\begin{aligned}
D(\mathfrak{s}+\mathfrak{t}) &= \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{s}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{s}+\mathfrak{t})\, d\mathfrak{a} \\
&= \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{s}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\big(\chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{s}) + \chi(\mathfrak{A},\mathfrak{a}+\mathfrak{s})\big)\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{s}+\mathfrak{t})\, d\mathfrak{a} \\
&\leqq \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{s}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{a})\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{s})\, d\mathfrak{a} + \int\limits_{[\mathfrak{s},\mathfrak{x}-\mathfrak{t}]} \chi(\mathfrak{A},\mathfrak{b})\, \chi(\overline{\mathfrak{A}},\mathfrak{b}+\mathfrak{t})\, d\mathfrak{b} \\
&\leqq D(\mathfrak{s}) + D(\mathfrak{t})
\end{aligned}
$$

16. Hilfssatz 4. 3. *Für alle* $\mathfrak{b} \in \mathfrak{B} \cap [\mathfrak{o}, \mathfrak{x}]$ *ist*

$$D_{\mathfrak{x}}(\mathfrak{b}) \leqq \underline{m}(\mathfrak{C},\mathfrak{x}) - m(\mathfrak{A},\mathfrak{x}). \tag{4. 12}$$

Beweis.

$$
\begin{aligned}
D_{\mathfrak{x}}(\mathfrak{b}) &= \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{b}]} \chi(\mathfrak{A},\mathfrak{a})\, \chi(\overline{\mathfrak{A}},\mathfrak{a}+\mathfrak{b})\, d\mathfrak{a} \\
&= \int\limits_{[\mathfrak{o},\mathfrak{x}-\mathfrak{b}]} \chi\big((\mathfrak{A}+\{\mathfrak{b}\}) - \mathfrak{A}, \mathfrak{a}+\mathfrak{b}\big)\, d\mathfrak{a} \\
&= \int\limits_{[\mathfrak{b},\mathfrak{x}]} \chi\big((\mathfrak{A}+\{\mathfrak{b}\}) - \mathfrak{A}, \mathfrak{c}\big)\, d\mathfrak{c} \\
&\leqq m\big((\mathfrak{A}+\{\mathfrak{b}\}) - \mathfrak{A}, \mathfrak{x}\big) = m\big(((\mathfrak{A}+\{\mathfrak{b}\}) \cup \mathfrak{A}) - \mathfrak{A}, \mathfrak{x}\big) \\
&= m\big((\mathfrak{A}+\{\mathfrak{b}\}) \cup \mathfrak{A}, \mathfrak{x}\big) - m(\mathfrak{A},\mathfrak{x}) \leqq \underline{m}(\mathfrak{C} \cup \mathfrak{A}, \mathfrak{x}) - m(\mathfrak{A},\mathfrak{x}) \\
&= \underline{m}(\mathfrak{C},\mathfrak{x}) - m(\mathfrak{A},\mathfrak{x});
\end{aligned}
$$

letzteres, da wegen $\mathfrak{o} \in \mathfrak{B}$ stets $\mathfrak{A} \subset \mathfrak{A}+\mathfrak{B} = \mathfrak{C}$ gilt.

Hilfssatz 4. 4. *Für alle* $\mathfrak{a} \in (\mathfrak{o}, \mathfrak{x})$ *gilt*

$$D(\mathfrak{a}) \leqq h(\mathfrak{a})\, \big(\underline{m}(\mathfrak{C},\mathfrak{x}) - m(\mathfrak{A},\mathfrak{x})\big). \tag{4. 13}$$

Beweis. Ist $\mathfrak{o} \leqq \mathfrak{c} \leqq \mathfrak{x}$ ein Punkt mit $g(\mathfrak{c}) < \infty$, also mit

$$\mathfrak{c} = \mathfrak{b}_1 + \cdots + \mathfrak{b}_{g(\mathfrak{c})};\ \mathfrak{b}_i \in \mathfrak{B},$$

so folgt aus den Hilfssätzen 4. 2 und 4. 3

$$
\begin{aligned}
D_{\mathfrak{x}}(\mathfrak{c}) &\leqq \sum_{i=1}^{g(\mathfrak{c})} D_{\mathfrak{x}}(\mathfrak{b}_i) \leqq \sum_{i=1}^{g} \big(\underline{m}(\mathfrak{C},\mathfrak{x}) - m(\mathfrak{A},\mathfrak{x})\big) \\
&= g(\mathfrak{c})\, \big(\underline{m}(\mathfrak{C},\mathfrak{x}) - m(\mathfrak{A},\mathfrak{x})\big).
\end{aligned}
$$

Für $g(\mathfrak{c}) = \infty$ gilt diese Ungleichung erst recht. Aus der Stetigkeit von $D_{\mathfrak{x}}(\mathfrak{m})$ folgt nun für alle $\mathfrak{a} \in (\mathfrak{o}, \mathfrak{x})$, die also im Inneren von $[\mathfrak{o}, \mathfrak{x}]$ liegen,

$$D_{\mathfrak{x}}^{-}(\mathfrak{a}) = \lim_{\mathfrak{c} \to \mathfrak{a}} D_{\mathfrak{x}}(\mathfrak{c}) \leqq (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) \varliminf_{\mathfrak{c} \to \mathfrak{a}} g(\mathfrak{c})$$
$$= (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) h(\mathfrak{a}).$$

Hilfssatz 4.5[14]. *Es gilt*

$$\forall \mathfrak{x} : (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) m(\mathfrak{x}) \cdot \lambda \geqq \int_{[\mathfrak{o}, \mathfrak{x}]} D_{\mathfrak{x}}(\mathfrak{a})\, d\mathfrak{a}, \tag{4.14}$$

ferner

$$\forall \delta > 0 \exists \mathfrak{r}_1(\delta) \forall \mathfrak{x} \geqq \mathfrak{r}_1 : \tag{4.15}$$
$$(\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) m(\mathfrak{x}) (\lambda^* + \delta) \geqq \int_{[\mathfrak{r}_1, \mathfrak{x}]} D_{\mathfrak{x}}(\mathfrak{a})\, d\mathfrak{a}.$$

Beweis. Man integriert (4.13) über seinen Gültigkeitsbereich $(\mathfrak{o}, \mathfrak{x})$ oder auch $[\mathfrak{o}, \mathfrak{x}]$, da beide Bereiche sich nur um eine Menge vom Maße 0 unterscheiden, und erhält

$$\int_{[\mathfrak{o}, \mathfrak{x}]} D_{\mathfrak{x}}(\mathfrak{a})\, d\mathfrak{a} \leqq (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) \int_{[\mathfrak{o}, \mathfrak{x}]} h(\mathfrak{a})\, d\mathfrak{a}$$
$$\leqq (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) m(\mathfrak{x}) \lambda$$

nach Definition von λ, also (4.14). Nach Definition von λ^* gilt

$$\forall \delta > 0 \exists \mathfrak{r}_1(\delta) \forall \mathfrak{x} \geqq \mathfrak{r}_1 : \int_{[\mathfrak{r}_1, \mathfrak{x}]} h(\mathfrak{a})\, d\mathfrak{a} \leqq (\lambda^* + \delta) m(\mathfrak{x}).$$

Integration von (4.13) über $[\mathfrak{r}_1, \mathfrak{x}]$ ergibt dann (4.15).

Hilfssatz 4.6[15]. *Bei beliebig vorgegebenen* $\mathfrak{y} \in \mathfrak{N}$ *gilt für alle* $\mathfrak{x} \geqq \mathfrak{y}$

$$\int_{[\mathfrak{y}, \mathfrak{x}]} D_{\mathfrak{x}}(\mathfrak{a})\, d\mathfrak{a} = \int_{[\mathfrak{y}, \mathfrak{x}]} m(\mathfrak{A}, \mathfrak{a} - \mathfrak{y}) \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}. \tag{4.16}$$

Beweis. Ich benutze den Integralsatz von Fubini[16]. $\int \cdots d(\mathfrak{a}, \mathfrak{b})$ bezeichnet das Integral im Produktraum $\mathfrak{N} \times \mathfrak{N}$.

$$\int_{[\mathfrak{y}, \mathfrak{x}]} D_{\mathfrak{x}}(\mathfrak{a})\, d\mathfrak{a} = \int_{[\mathfrak{y}, \mathfrak{x}]} d\mathfrak{a} \int_{[\mathfrak{o}, \mathfrak{x} - \mathfrak{a}]} \chi(\mathfrak{A}, \mathfrak{b}) \chi(\overline{\mathfrak{A}}, \mathfrak{a} + \mathfrak{b})\, d\mathfrak{b}$$
$$= \int_{\substack{\mathfrak{y} \leq \mathfrak{a} \leq \mathfrak{x} \\ \mathfrak{y} \leq \mathfrak{a} + \mathfrak{b} \leq \mathfrak{x}}} \chi(\mathfrak{A}, \mathfrak{b}) \chi(\overline{\mathfrak{A}}, \mathfrak{a} + \mathfrak{b})\, d(\mathfrak{a}, \mathfrak{b}) = \int_{\mathfrak{y} \leq \mathfrak{y} + \mathfrak{b} \leq \mathfrak{x}} \chi(\mathfrak{A}, \mathfrak{b}) \chi(\mathfrak{A}, \overline{\mathfrak{c}})\, d(\mathfrak{c}, \mathfrak{b})$$
$$= \int_{[\mathfrak{y}, \mathfrak{x}]} \chi(\overline{\mathfrak{A}}, \mathfrak{c}) \int_{[\mathfrak{o}, \mathfrak{c} - \mathfrak{y}]} \chi(\mathfrak{A}, \mathfrak{b})\, d\mathfrak{b}\, d\mathfrak{c} = \int_{[\mathfrak{y}, \mathfrak{x}]} \chi(\overline{\mathfrak{A}}, \mathfrak{c}) m(\mathfrak{A}, \mathfrak{c} - \mathfrak{y})\, d\mathfrak{c}.$$

17. Satz 4.3. *Es gelten die Abschätzungen*

$$\text{(a)} \quad m(\mathfrak{A}, \mathfrak{x}) \geqq \alpha m(\mathfrak{x}) \tag{4.17}$$
$$\text{(b)} \quad (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) m(\mathfrak{x}) \lambda \geqq \alpha \int_{[\mathfrak{o}, \mathfrak{x}]} m(\mathfrak{a}) \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}$$

---

[14]) Von diesen Formeln ausgehend, beweist Chatrovsky seinen Satz. Unsere Beweisführung bis hierher wich zum Teil von der seinen ab und lehnte sich auch an die Arbeit von Schell [18] an.

[15]) Eine entsprechende Umformung mit Summen statt Integralen nimmt Kasch in seiner Arbeit [7] über Gitterpunktmengen vor.

[16]) Vgl. Halmos [4], § 36.

für alle $\mathfrak{x}$, *ferner*

(4. 18) $$\forall \delta, \varepsilon > 0 \exists \mathfrak{r}(\varepsilon, \delta) \forall \mathfrak{x} \geqq \mathfrak{r}:$$

(a$_1$) $m(\mathfrak{A}, \mathfrak{x}) \geqq (\alpha^* - \varepsilon)\, m(\mathfrak{x})$,

(a$_2$) $m(\mathfrak{A}, \mathfrak{x}) \leqq (\alpha^* + \varepsilon)\, m(\mathfrak{x})$, *falls die natürliche Dichte von* $\mathfrak{A}$ *existiert*,

(b) $((\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x}))\, m(\mathfrak{x})\, (\lambda^* + \delta) \geqq (\alpha^* - \varepsilon) \int\limits_{[\mathfrak{r}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}$.

Beweis. (4. 17a) gilt nach Definition von α. Setzt man dies mit $\mathfrak{x} = \mathfrak{a}$ und (4. 16) mit $\mathfrak{y} = 0$ in (4. 14) ein, so erhält man (4. 17b).

Nach Definition von α^* gibt es jedenfalls zu jedem $\varepsilon > 0$ ein $\mathfrak{r}_2(\varepsilon)$, so daß (4. 18a$_1$) und gegebenenfalls auch (4. 18a$_2$) für alle $\mathfrak{x} \geqq \mathfrak{r}_2$ gelten. Außerdem gibt es zu jedem $\varepsilon > 0$ und jedem $\mathfrak{r}_1(\delta)$, das gemäß (4. 13) bestimmt sei, ein $\mathfrak{r}_3(\varepsilon, \delta)$, so daß $m(\mathfrak{x} - \mathfrak{r}_1) \geqq (1 - \varepsilon)\, m(\mathfrak{x})$ für $\mathfrak{x} \geqq \mathfrak{r}_3$ wird. Nun wähle ich

(4. 19) $$\mathfrak{r}(\varepsilon, \delta) \geqq \mathfrak{r}_1(\delta) + \mathfrak{r}_2\left(\frac{\varepsilon}{2}\right) \text{ und } \geqq \mathfrak{r}_3\left(\frac{\varepsilon}{2}, \delta\right).$$

Dann gilt für alle $\mathfrak{x} \geqq \mathfrak{r}(\varepsilon, \delta)$ zunächst (4. 15). Durch Anwendung von (4. 16) mit $\mathfrak{y} = \mathfrak{r}_1$ und von (4. 18a$_1$) für $\mathfrak{x} \geqq \mathfrak{r}_2$ folgt sodann

$$\begin{aligned}
(\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x}))\, m(\mathfrak{x})\, (\lambda^* + \delta) &\geqq \int\limits_{[\mathfrak{r}_1, \mathfrak{x}]} m(\mathfrak{A}, \mathfrak{a} - \mathfrak{r}_1)\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} \\
&\geqq \int\limits_{[\mathfrak{r}, \mathfrak{x}]} m(\mathfrak{A}, \mathfrak{a} - \mathfrak{r}_1)\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} \\
&\geqq \left(\alpha^* - \frac{\varepsilon}{2}\right) \int\limits_{[\mathfrak{r}, \mathfrak{x}]} m(\mathfrak{a} - \mathfrak{r}_1)\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} \\
&\geqq \left(\alpha^* - \frac{\varepsilon}{2}\right)\left(1 - \frac{\varepsilon}{2}\right) \int\limits_{[\mathfrak{r}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} \\
&\geqq (\alpha^* - \varepsilon) \int\limits_{[\mathfrak{r}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a},
\end{aligned}$$

also (4. 18b). Die Formeln (4. 18a) gelten schon für $\mathfrak{x} \geqq \mathfrak{r}_2(\varepsilon)$, also erst recht für $\mathfrak{x} \geqq \mathfrak{r}(\varepsilon, \delta)$.

Es ist für das Folgende praktisch, die beiden Formeln (4. 17) und (4. 18) zu einer einzigen zusammenzufassen.

Hauptformel (Satz 4. 3). *Setzt man entweder* $a = \alpha$, $l = \lambda$, $\mathfrak{t} = \mathfrak{o}$ *oder, falls* $\varepsilon, \delta > 0$ *beliebig vorgegeben sind,* $a = \alpha^* - \varepsilon$, $l = \lambda^* + \delta$, $\mathfrak{t} = \mathfrak{r}(\varepsilon, \delta)$, *so gilt für alle* $\mathfrak{x} \geqq \mathfrak{t}$

(4. 20) (a) $m(\mathfrak{A}, \mathfrak{x}) \geqq a\, m(\mathfrak{x})$,

(b) $(\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x}))\, m(\mathfrak{x})\, l \geqq a \int\limits_{[\mathfrak{t}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}$.

§ 5. Beweis der Sätze 4. 1 und 4. 3.

18. Ich schätze jetzt das Integral in der Hauptformel weiter ab, indem ich das Integrationsgebiet $[\mathfrak{t}, \mathfrak{x}]$ zerteile. Ich wähle beliebige aber feste Zahlen $0 = \varrho_0 < \varrho_1 < \cdots < \varrho_{n-1} < \varrho_n = 1$ und betrachte die Hyperebenen [17])

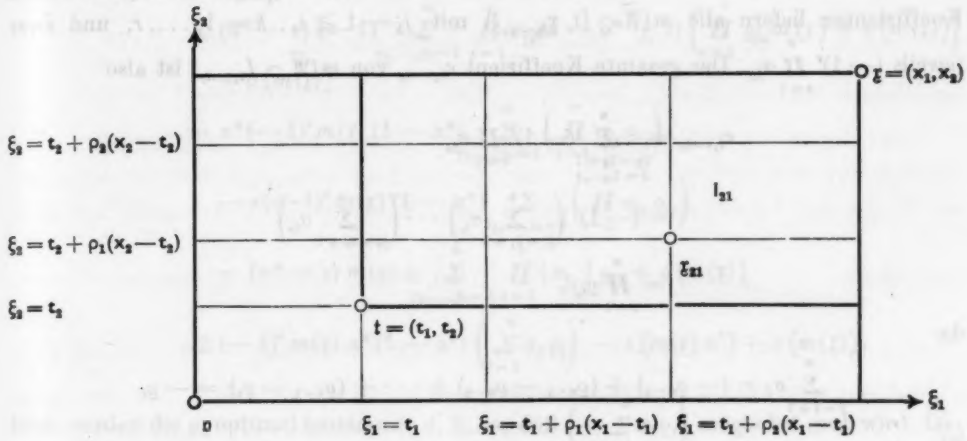
$$\xi_i = \varrho_k(x_i - t_i) + t_i; \quad i = 1, \ldots, r; \quad k = 0, \ldots, n.$$

[17]) Man könnte eine noch allgemeinere Einteilung wählen, indem man auf den einzelnen Achsen verschiedene und auch verschieden viele Teilpunkte ϱ benutzt. Die Überlegungen lassen sich dann analog zu den folgenden führen; nur werden die Formeln wesentlich komplizierter, während sich das Ergebnis nicht verbessert.

Den Schnittpunkt von

$$\xi_1 = \varrho_{i_1}(x_1 - t_1) + t_1, \ldots, \xi_r = \varrho_{i_r}(x_r - t_r) + t_r$$

nenne ich $\mathfrak{x}_{i_1 \ldots i_r}$. Speziell ist $\mathfrak{x}_{0 \ldots 0} = \mathfrak{t}$ und $\mathfrak{x}_{n \ldots n} = \mathfrak{x}$. Das Intervall $[\mathfrak{x}_{i_1 \ldots i_r}, \mathfrak{x}_{i_1+1 \ldots i_r+1}]$ heiße $I_{i_1 \ldots i_r}$.



Dann gilt $[\mathfrak{t}, \mathfrak{x}] = \bigcup_{i_1, \ldots, i_r = 0}^{n-1} I_{i_1 \ldots i_r}$, und zwei verschiedene $I \ldots$ haben höchstens einen Teil ihres Randes, also eine Menge vom Maße 0, gemeinsam. Folglich gilt

$$(5.1) \quad \int_{[\mathfrak{t}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} = \sum_{i_1, \ldots, i_r = 0}^{n-1} \int_{I_{i_1 \ldots i_r}} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}$$
$$\geqq \sum_{i_1, \ldots, i_r = 0}^{n-1} m(\mathfrak{x}_{i_1 \ldots i_r}) \int_{I_{i_1 \ldots i_r}} \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a}.$$

Nun ist

$$(5.2) \quad m(\mathfrak{x}_{i_1 \ldots i_r}) = m([\mathfrak{t}, \mathfrak{x}_{i_1 \ldots i_r}]) + m([\mathfrak{o}, \mathfrak{x}_{i_1 \ldots i_r}] - [\mathfrak{t}, \mathfrak{x}_{i_1 \ldots i_r}])$$
$$= \prod_{s=1}^{r} \varrho_{i_s} m([\mathfrak{t}, \mathfrak{x}]) + o(m(\mathfrak{x}))$$
$$= \prod_{s=1}^{r} \varrho_{i_s} m(\mathfrak{x}) + o(m(\mathfrak{x}))$$

für $\mathfrak{x} \to \infty$ bei festem $\mathfrak{t}$. Für $\mathfrak{t} = \mathfrak{o}$ fällt $o(m(\mathfrak{x}))$ ganz weg. (5.1) und (5.2) zusammen ergeben

$$(5.3) \quad \int_{[\mathfrak{t}, \mathfrak{x}]} m(\mathfrak{a})\, \chi(\overline{\mathfrak{A}}, \mathfrak{a})\, d\mathfrak{a} \geqq m(\mathfrak{x})\,(1 + o(1)) \left(\sum_{i_1 \ldots i_r = 0}^{n-1} \prod_{s=1}^{r} \varrho_{i_s} m(\overline{\mathfrak{A}} \cap I_{i_1 \ldots i_r}) \right).$$

19. Hilfssatz 5. 1. *Es gilt die Umformung*

$$(5.4) \quad \sum_{i_1, \ldots, r = 0}^{n-1} \prod_{s=1}^{r} \varrho_{i_s} m(\overline{\mathfrak{A}} \cup I_{i_1 \ldots i_r}) = (-1)^r \sum_{j_1, \ldots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s} m(\overline{\mathfrak{A}} \cap [\mathfrak{t}, \mathfrak{x}_{j_1 \ldots j_r}])$$

mit $\sigma_i = \varrho_i - \varrho_{i-1}$, $i = 1, \ldots, n-1$, *und* $\sigma_n = -\varrho_{n-1}$.

Beweis. Auf der rechten Seite zerlege man wieder

$$m(\overline{\mathfrak{A}} \cap [\mathfrak{t}, \mathfrak{x}_{j_1 \dots j_r}]) = \sum_{\substack{i_k = 0 \\ k = 1, \dots, r}}^{j_k - 1} m(\overline{\mathfrak{A}} \cap I_{i_1 \dots i_r})$$

und ordne dann die rechte Summe in (5. 4) nach den $m(\overline{\mathfrak{A}} \cap I_{i_1 \dots i_r})$. Beiträge zu dessen Koeffizienten liefern alle $m(\overline{\mathfrak{A}} \cap [\mathfrak{t}, \mathfrak{x}_{j_1 \dots j_r}])$ mit $j_k - 1 \geqq i_k$, $k = 1, \dots, r$, und zwar jeweils $(-1)^r \prod_{s=1}^{r} \sigma_{j_s}$. Der gesamte Koeffizient $c_{i_1 \dots i_r}$ von $m(\overline{\mathfrak{A}} \cap I_{i_1 \dots i_r})$ ist also

$$\begin{aligned} c_{i_1 \dots i_r} &= \sum_{\substack{j_k = i_k + 1 \\ k = 1, \dots, r}}^{n} (-1)^r \prod_{s=1}^{r} \sigma_{j_s} \\ &= (-1)^r \left(\sum_{j_1 = i_1 + 1}^{n} \sigma_{j_1} \right) \cdots \left(\sum_{j_r = i_r + 1}^{n} \sigma_{j_r} \right) \\ &= \prod_{s=1}^{n} \varrho_{i_s}, \end{aligned}$$

da

$$\sum_{j=i+1}^{n} \sigma_j = (-\varrho_{n-1}) + (\varrho_{n-1} - \varrho_{n-2}) + \cdots + (\varrho_{i+1} - \varrho_i) = -\varrho_i$$

ist.

Außerdem gilt

$$\begin{aligned} (5.5) \qquad & m(\overline{\mathfrak{A}} \cap [\mathfrak{t}, \mathfrak{x}_{j_1 \dots j_r}]) \\ &= m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \dots j_r}) - m(\overline{\mathfrak{A}} \cap ([\mathfrak{o}, \mathfrak{x}_{j_1 \dots j_r}] - [\mathfrak{t}, \mathfrak{x}_{j_1 \dots j_r}])) \\ &= m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \dots j_r}) + o(m(\mathfrak{x})), \end{aligned}$$

wieder für $\mathfrak{x} \to \infty$ bei festem $\mathfrak{t}$. Für $\mathfrak{t} = \mathfrak{o}$ fällt auch hier $o(m(\mathfrak{x}))$ weg.

Setzt man nun sukzessive (5. 5) in (5. 4) in (5. 3) in (4. 20b) ein, so folgt

$$\begin{aligned} \forall \mathfrak{x} \geqq \mathfrak{t}: \ & (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) m(\mathfrak{x}) \, l \\ & \geqq a m(\mathfrak{x}) (1 + o(1)) (-1)^r \sum_{j_1, \dots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s} (m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \dots j_r}) + o(m(\mathfrak{x}))), \end{aligned}$$

also

$$\begin{aligned} (5.6) \qquad \forall \mathfrak{x} \geqq \mathfrak{t}: \ & (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) \, l \\ & \geqq a (-1)^r \sum_{j_1, \dots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s} m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \dots j_r}) + o(m(\mathfrak{x})) \end{aligned}$$

für $\mathfrak{x} \to \infty$, festes $\mathfrak{t}$ und festes n. Für $\mathfrak{t} = \mathfrak{o}$ fehlt das o-Glied ganz.

20. *Beweis von Satz* 4. 3. Nach Voraussetzung existiert die natürliche Dichte α^* von $\mathfrak{A}$. Wählt man beliebige aber feste $\varepsilon, \delta > 0$, so folgt aus (4. 18) und (5. 6) mit $\mathfrak{t} = \mathfrak{r}(\varepsilon, \delta)$

$\forall \mathfrak{x} \geqq \mathfrak{r}(\varepsilon, \delta)$:

$$\text{(a)} \qquad (\alpha^* - \varepsilon) \, m(\mathfrak{x}) \leqq m(\mathfrak{A}, \mathfrak{x}) \leqq (\alpha^* + \varepsilon) \, m(\mathfrak{x}),$$

$$\begin{aligned} \text{(b)} \qquad & (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x})) \, (\lambda^* + \delta) \\ & \geqq (\alpha^* - \varepsilon) (-1)^r \sum_{j_1, \dots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s} m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \dots j_r}) + o(m(\mathfrak{x})). \end{aligned}$$

Die Voraussetzung $\mathfrak{x} \geqq \mathfrak{r}$ ist für alle $\mathfrak{x}_{j_1 \ldots j_r}$ erfüllt. Man kann also $m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \ldots j_r})$ nach (a) abschätzen. Beachtet man $\delta^*(\overline{\mathfrak{A}}) = 1 - \delta^*(\mathfrak{A})$, (5. 2), sowie das Vorzeichen der Koeffizienten, so folgt aus (b)

$$
\begin{aligned}
(5.7)\quad & (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x}))(\lambda^* + \delta) \\
& \geqq (\alpha^* - \varepsilon)(-1)^r \sum_{j_1, \ldots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s}(1 - \alpha^* \pm \varepsilon) \left(\prod_{s=1}^{r} \varrho_{j_s} m(\mathfrak{x}) + o(m(\mathfrak{x})) \right) \\
& \quad + o(m(\mathfrak{x})) \\
& = \alpha^*(-1)^r m(\mathfrak{x})(1 - \alpha^*) \sum_{j_1, \ldots, j_r = 1}^{n} \left(\prod_{s=1}^{r} \sigma_{j_s} \varrho_{j_s} \right) \\
& \quad - \varepsilon(-1)^r m(\mathfrak{x})(1 - \alpha^*) \sum_{j_1, \ldots, j_r = 1}^{n} \left(\prod_{s=1}^{r} \sigma_{j_s} \varrho_{j_s} \right) \\
& \quad - (\alpha^* - \varepsilon) m(\mathfrak{x}) \varepsilon \sum_{j_1, \ldots, j_r = 1}^{n} \prod_{s=1}^{r} |\sigma_{j_s}| \varrho_{j_s} + o(m(\mathfrak{x})) \\
& \geqq (-1)^r m(\mathfrak{x}) \alpha^*(1 - \alpha^*) \left(\sum_{j=1}^{n} \sigma_j \varrho_j \right)^r - \varepsilon (m(\mathfrak{x}) n^r) + o(m(\mathfrak{x})).
\end{aligned}
$$

Jetzt werden die ϱ_i optimal bestimmt, d. h. so, daß $\left(-\sum_{j=1}^{n} \sigma_j \varrho_j\right)$ möglichst groß wird. Dazu bildet man für $\nu = 1, \ldots, n-1$

$$
0 = \frac{\partial}{\partial \varrho_\nu} \left(-\sum_{j=1}^{n} \sigma_j \varrho_j \right) = \varrho_{\nu-1} + \varrho_{\nu+1} - 2\varrho_\nu .
$$

Unter Benutzung von $\varrho_0 = 0$, $\varrho_n = 1$ folgt hieraus eindeutig

$$
\varrho_\nu = \frac{\nu}{n}, \qquad \nu = 0, \ldots, n
$$

und

$$
\left(-\sum_{j=1}^{n} \sigma_j \varrho_j \right) = \frac{1}{2} \frac{n-1}{n} \text{ [18]}.
$$

Hiermit ergibt sich aus (5. 7)

$$
\begin{aligned}
(5.8)\quad & \forall \mathfrak{x} \geqq \mathfrak{r}(\varepsilon, \delta): (\underline{m}(\mathfrak{C}, \mathfrak{x}) - m(\mathfrak{A}, \mathfrak{x}))(\lambda^* + \delta) \\
& \geqq m(\mathfrak{x}) \alpha^*(1 - \alpha^*) \frac{1}{2^r} \left(\frac{n-1}{n} \right)^r - \varepsilon m(\mathfrak{x}) n^r + o(m(\mathfrak{x})),
\end{aligned}
$$

also

$$
\frac{\underline{m}(\mathfrak{C}, \mathfrak{x})}{m(\mathfrak{x})} \geqq (\alpha^* - \varepsilon) + \frac{\alpha^*(1 - \alpha^*)}{2^r(\lambda^* + \delta)} \left(\frac{n-1}{n} \right)^r - \varepsilon \frac{n^r}{\lambda^* + \delta} + o(1)
$$

und hieraus für $\mathfrak{x} \to \infty$

$$
\gamma^* \geqq \alpha^* \left(1 + \frac{1 - \alpha^*}{2^r(\lambda^* + \delta)} \left(\frac{n-1}{n} \right)^r \right) - \varepsilon \left(\frac{n^r}{\lambda^* + \delta} + 1 \right).
$$

[18]) Daß dies wirklich ein Maximum ist, folgt aus detaillierteren Betrachtungen, auf die ich nicht eingehe.

Da ε, δ beliebig waren, folgt weiter

$$\gamma^* \geqq \alpha^* \left(1 + \frac{1-\alpha^*}{2^r \lambda^*} \left(\frac{n-1}{n}\right)^r\right)$$

und wegen der Willkürlichkeit von n schließlich mit $n \to \infty$

$$\gamma^* \geqq \alpha^* \left(1 + \frac{1-\alpha^*}{2^r \lambda^*}\right),$$

also Satz 4. 3.

21. *Beweis von Satz* 4. 1. Ausgangspunkt ist auch jetzt die Ungleichung (5. 6). Zur Abschätzung von $m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \ldots j_r})$ steht aber nur die Ungleichung (4. 20 a) für alle $\mathfrak{x} \geqq \mathfrak{t}$ zur Verfügung. Angewandt auf $\overline{\mathfrak{A}}$ statt $\mathfrak{A}$ lautet sie unter Berücksichtigung von (5. 2)

$$m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \ldots j_r}) \leqq (1-a) \prod_{s=1}^{r} \varrho_{j_s} m(\mathfrak{x}) + o(m(\mathfrak{x})). \tag{5. 9}$$

Hiermit können in (5. 6) nur die Glieder mit negativem Koeffizienten $(-1)^r \prod_{s=1}^{r} \sigma_{j_s} \cdot a$ abgeschätzt werden. Die positiven ersetzt man trivialerweise durch 0. Nur das immer positive Glied für $j_1 = \cdots = j_r = n$, das $a \varrho_{n-1}^r m(\overline{\mathfrak{A}}, \mathfrak{x})$ lautet, kann nichttrivial abgeschätzt werden, da von der linken Seite

$$l m(\mathfrak{A}, \mathfrak{x}) = l m(\mathfrak{x}) - l m(\overline{\mathfrak{A}}, \mathfrak{x})$$

hinzukommt und $a \varrho_{n-1}^r - l < 0$ ist.

Nach Definition in (5. 4) ist $\sigma_j > 0$ für $j = 1, \ldots, n-1$ und $\sigma_n < 0$. Folglich hängt das Vorzeichen von $(-1)^r a \prod_{s=1}^{r} \sigma_{j_s}$ außer von $(-1)^r$ von der Anzahl der $j_s = n$ ab. Daher ordne ich die Summe in (5. 6) zunächst nach diesen. Wie leicht zu sehen, gilt

$$\begin{aligned} &\sum_{j_1, \ldots, j_r = 1}^{n} \prod_{s=1}^{r} \sigma_{j_s} m(\overline{\mathfrak{A}}, \mathfrak{x}_{j_1 \ldots j_r}) \\ &= \sum_{\varkappa=0}^{r} (-\varrho_{n-1})^{\varkappa} \sum_{k_1, \ldots, k_{r-\varkappa} = 1}^{n-1} \sigma_{k_1} \cdots \sigma_{k_{r-\varkappa}} \sum_{\substack{\varkappa\text{-mal } n \\ \text{sonst } k_1, \ldots, k_{r-\varkappa}}} m(\overline{\mathfrak{A}}, \mathfrak{x} \ldots). \end{aligned} \tag{5. 10}$$

Die letzte Summationsvorschrift ist so gemeint, daß über alle Indexsysteme in $\mathfrak{x} \ldots$ summiert wird, in denen $\varkappa$-mal n auftritt $\left(\binom{r}{\varkappa}\right.$ Möglichkeiten$\left.\right)$ und die freigebliebenen Stellen mit $k_1, \ldots, k_{r-\varkappa}$ (in dieser Reihenfolge) besetzt sind. (Letztere Besetzung ist zwangsläufig.)

Setze ich nun

$$A = \sum_{\mu=0}^{\left[\frac{r}{2}\right]} \varrho_{n-1}^{2\mu} \sum_{k_1, \ldots, k_{r-2\mu} = 1}^{n-1} \sigma_{k_1} \cdots \sigma_{k_{r-2\mu}} \sum_{\substack{2\mu \text{ mal } n \\ \text{sonst } k_1, \ldots, k_{r-2\mu}}} m(\overline{\mathfrak{A}}, \mathfrak{x} \ldots) \tag{5. 11}$$

$$B = \sum_{\mu=0}^{\left[\frac{r-1}{2}\right]} \varrho_{n-1}^{2\mu+1} \sum_{k_1, \ldots, k_{r-2\mu-1} = 1}^{n-1} \sigma_{k_1} \cdots \sigma_{k_{r-2\mu-1}} \sum_{\substack{2\mu+1 \text{ mal } n \\ \text{sonst } k_1, \ldots, k_{r-2\mu-1}}} m(\overline{\mathfrak{A}}, \mathfrak{x} \ldots) \tag{5. 12}$$

so treten in A und B nur noch positive Koeffizienten auf und aus (5. 6) wird

$$(5.\,13)\quad \forall \mathfrak{x} \geqq \mathfrak{t}\colon \underline{m}(\mathfrak{C}, \mathfrak{x})\, l \geqq l m(\mathfrak{x}) - l m(\overline{\mathfrak{A}}, \mathfrak{x}) + a(-1)^r (A - B) + o\,(m(\mathfrak{x})).$$

Ich trenne jetzt vorübergehend die Fälle mit geradem und ungeradem r und vernachlässige rechts alle nicht abschätzbaren positiven Glieder.

$$(5.\,14\mathrm{a})\quad r \text{ gerade}\colon \underline{m}(\mathfrak{C}, \mathfrak{x})\, l \geqq m(\mathfrak{x})\, l - (l - a\varrho_{n-1}^r)\, m(\overline{\mathfrak{A}}, \mathfrak{x}) - aB + o\,(m(\mathfrak{x})),$$

$$(5.\,14\mathrm{b})\quad r \text{ ungerade}\colon \underline{m}(\mathfrak{C}, \mathfrak{x})\, l \geqq m(\mathfrak{x})\, l - (l - a\varrho_{n-1}^r)\, m(\overline{\mathfrak{A}}, \mathfrak{x}) - aA + o\,(m(\mathfrak{x})).$$

Jetzt kann ich durchweg (5. 9) verwenden und erhalte

$$(5.\,15\mathrm{a})\quad \underline{m}(\mathfrak{C}, \mathfrak{x})\, l \geqq m(\mathfrak{x})\, l - (l - a\varrho_{n-1}^r)\,(1 - a)\, m(\mathfrak{x})$$

$$- a(1 - a)\, m(\mathfrak{x}) \sum_{\mu=0}^{\left[\frac{r-1}{2}\right]} \varrho_{n-1}^{2\mu+1} \sum_{k_1, \ldots, k_{r-2\mu-1}=1}^{n-1} \binom{r}{2\mu+1} \sigma_{k_1} \varrho_{k_1} \cdots \sigma_{k_{r-2\mu-1}} \varrho_{k_{r-2\mu-1}} + o\,(m(\mathfrak{x})),$$

$$(5.\,15\mathrm{b})\quad \underline{m}(\mathfrak{C}, \mathfrak{x})\, l \geqq m(\mathfrak{x})\, l - (l - a\varrho_{n-1}^r)\,(1 - a)\, m(\mathfrak{x})$$

$$- a(1 - a)\, m(\mathfrak{x}) \sum_{\mu=0}^{\left[\frac{r}{2}\right]} \varrho_{n-1}^{2\mu} \sum_{k_1, \ldots, k_{r-2\mu}=1}^{n-1} \binom{r}{2\mu} \sigma_{k_1} \varrho_{k_1} \cdots \sigma_{k_{r-2\mu}} \varrho_{k_{r-2\mu}} + o\,(m(\mathfrak{x})).$$

Das läßt sich auf die Form

$$(5\,16)\quad \forall \mathfrak{x} \geqq \mathfrak{t}\colon \frac{\underline{m}\,(\mathfrak{C}, \mathfrak{x})}{m(\mathfrak{x})} \geqq a\left(1 + \frac{1 - a}{l} f(r, n)\right) + o(1)$$

bringen, wobei man für $f(r, n)$ zunächst die Ausdrücke

$$(5.\,17\mathrm{a})\quad f_{\mathrm{a}}(r, n) = \varrho_{n-1}^r - \sum_{\mu=0}^{\left[\frac{r-1}{2}\right]} \binom{r}{2\mu+1} \varrho_{n-1}^{2\mu+1} \sum_{k_1, \ldots, k_{r-2\mu-1}=1}^{n-1} \prod_{s=1}^{r-2\mu-1} \sigma_{k_s} \varrho_{k_s}$$

$$(5.\,17\mathrm{b})\quad f_{\mathrm{b}}(r, n) = \varrho_{n-1}^r - \sum_{\mu=0}^{\left[\frac{r}{2}\right]} \binom{r}{2\mu} \varrho_{n-1}^{2\mu} \sum_{k_1, \ldots, k_{r-2\mu}=1}^{n-1} \prod_{s=1}^{r-2\mu} \sigma_{k_s} \varrho_{k_s}$$

erhält. Durch die Substituiion $2\mu + 1 = r - 2\mu' - 1$ bzw. $2\mu = r - 2\mu' - 1$ ergibt sich hieraus die gemeinsame Formel

$$(5.\,18)\quad f(r, n) = \varrho_{n-1}^r - \sum_u \binom{r}{u} \varrho_{n-1}^{r-u} \left(\sum_{k=1}^{n-1} \sigma_k \varrho_k\right)^u.$$

Dabei soll $\sum\limits_u$ wie später $\sum\limits_g$ immer die Summe über alle ungeraden bzw. geraden nichtnegativen ganzen Zahlen bedeuten. Man kann formal über alle u summieren, da $\binom{r}{u} = 0$ für $u > r$ ist.

Aus (5. 16) folgt im finiten Fall ($a = \alpha$, $l = \lambda$, $\mathfrak{t} = \mathfrak{o}$, $o(1) = 0$)

$$(5.\,19)\quad \gamma \geqq \alpha\left(1 + \frac{1 - \alpha}{\lambda} f(r, n)\right)$$

und im asymptotischen $(a = \alpha^* - \varepsilon,\ l = \lambda^* + \delta,\ \mathfrak{t} = \mathfrak{r}(\varepsilon, \delta))$

$$\gamma^* \geqq (\alpha^* - \varepsilon)\left(1 + \frac{(1 - \alpha^* + \varepsilon)}{\lambda^* + \delta} f(r, n)\right),$$

also auch

$$\gamma^* \geqq \alpha^*\left(1 + \frac{1 - \alpha^*}{\lambda^*} f(r, n)\right). \tag{5. 20}$$

Das sind die Aussagen (4. 3) und (4. 4) von Satz 4. 1 bis auf den Nachweis, daß $f(r, n)$ durch das dort angegebene $F(r)$ ersetzt werden kann. Das soll jetzt noch gezeigt werden.

22. Die ϱ_ν waren bisher bis auf die Bedingung $0 = \varrho_0 < \varrho_1 < \cdots < \varrho_n = 1$ noch frei wählbar. Sie werden jetzt durch Maximumbetrachtungen festgelegt. Dazu haben wir die Gleichungen

$$\frac{\partial f(r, n)}{\partial \varrho_\nu} = 0;\ \nu = 1, \ldots, n - 1$$

zu untersuchen. Für $\nu = 1, \ldots, n - 2$ bekommt man

$$\frac{\partial f(r, n)}{\partial \varrho_\nu} = -\sum_u \binom{r}{u} \varrho_{n-1}^{r-u}\, u \left(\sum_{k=1}^{n-1} \sigma_k \varrho_k\right)^{u-1} (2\varrho_\nu - \varrho_{\nu-1} - \varrho_{\nu+1}).$$

Der Koeffizient von $(2\varrho_\nu - \varrho_{\nu-1} - \varrho_{\nu+1})$ ist < 0, so daß

$$2\varrho_\nu - \varrho_{\nu-1} - \varrho_{\nu+1} = 0;\ \nu = 1, \ldots, n - 2$$

und hieraus eindeutig

$$\varrho_\nu = \nu\varrho_1;\ \nu = 1, \ldots, n - 1 \tag{5. 21}$$

folgt. Setzt man dies in (5. 20) ein und substituiert noch $x = \frac{\varrho_1 n}{2}$, so entsteht

$$f(r, n) = \left(\frac{n - 1}{n}\right)^r 2^r x^r \left(1 - \sum_u \binom{r}{u} x^u\right). \tag{5. 22}$$

Da (5. 19) und (5. 20) für alle n gelten, kann man $f(r, n)$ durch $\lim\limits_{n \to \infty} f(r, n) = F(r)$ ersetzen, wobei

$$F(r) = 2^r x^r \left(1 - \sum_u \binom{r}{u} x^u\right) \tag{5. 23}$$

ist. Das ist die Gleichung (4. 5) aus Satz 4. 1. Die Gleichung (4. 6) für den optimalen Wert $x = x_0$ erhält man schließlich aus $\frac{dF(r)}{dx} = 0$, was der noch nicht ausgenutzten Bedingung $\frac{\partial f}{\partial \varrho_{n-1}} = 0$ entspricht.

Daß die reelle Lösung der Gleichung

$$\sum_u \binom{r}{u} (r + u)\, x^u = r$$

eindeutig bestimmt ist, folgt aus der Monotonie der linken Seite.

Die Abschätzungen

$$0 < r = \sum_u \binom{r}{u} (r+u)\, x_0^u \qquad \text{und} \qquad r \leqq \sum_u \binom{r}{u} (r+u) \left(\frac{1}{r+1}\right)^u$$

ergeben, ebenfalls auf Grund der Monotonie,

$$0 < x_0 \leqq \frac{1}{r+1}.$$

Daher gilt

$$0 < \frac{2x_0}{n} = \varrho_1 < (n-1)\,\varrho_1 = \varrho_{n-1} \leqq \frac{n-1}{n}\,\frac{2}{r+1} < 1,$$

so daß die Bedingung $0 < \varrho_1 < \cdots < \varrho_{n-1} < 1$ für unsere Wahl der ϱ_i erfüllt ist.

Damit ist Satz 4. 1 bewiesen.

§ 6. Die Abschätzungen für $F(r)$.

23. Nunmehr werden für $F(r)$ die Aussagen von Satz 4. 2 hergeleitet. Bisher wurde $F(r)$ zu

$$F(r) = 2^r x_0^r \left(1 - \sum_u \binom{r}{u} x_0^u\right) \tag{6. 1}$$

bestimmt, wobei x_0 die einzige reelle Lösung von

$$\sum_u \binom{r}{u} (r+u)\, x_0^u = r \tag{6. 2}$$

bedeutet.

Satz 6. 1. *Für die Lösung x_0 von* (6. 2) *gilt*

$$\frac{w_0}{r+w_0} \leqq x_0 \leqq \frac{w_0}{r+v_0} \tag{6. 3}$$

mit $w_0 = ln(1+\sqrt{2}) = 0{,}8814\ldots$ *und* $v_0 = \dfrac{1}{2\sqrt{2}}\,(2 - w_0) = 0{,}3955\ldots.$

Beweis. Zunächst ist $w_0 = ln\,(1+\sqrt{2})$ die einzige reelle Lösung von

$$\sum_u \frac{w^u}{u!} = \mathfrak{Sin}\, w = 1,$$

wie man mit Hilfe der Darstellung $\mathfrak{Sin}\, w = \dfrac{1}{2}\,(e^w - e^{-w})$ einsieht. Es gilt also

$$\sum_u \frac{w_0^u}{u!} = \mathfrak{Sin}\, w_0 = 1 \qquad \text{und} \qquad \sum_g \frac{w_0^g}{g!} = \mathfrak{Cos}\, w_0 = \sqrt{1 + \mathfrak{Sin}^2\, w_0} = \sqrt{2}. \tag{6. 4}$$

Weiter zeigt man durch Induktion, daß für alle ganzen $n \geqq 0$

$$\binom{r}{n} \geqq \left(1 - \frac{n(n-1)}{2r}\right) \frac{r^n}{n!} \tag{6. 5}$$

und

$$\left(\frac{r}{r+v_0}\right)^n \geqq 1 - \frac{nv_0}{r+v_0}, \tag{6. 6}$$

sowie für alle ganzen $n \geqq 3$

$$\binom{r}{n} (r+n) \leqq \frac{r^{n+1}}{n!} \tag{6. 7}$$

ist.

Zum Nachweis von (6. 3) genügt es nun wegen der Monotonie der linken Seite von (6. 2),

$$(6.\,8)\quad \frac{1}{r}\sum_u \binom{r}{u}(r+u)\left(\frac{w_0}{r+w_0}\right)^u \leqq 1 \leqq \frac{1}{r}\sum_u \binom{r}{u}(r+u)\left(\frac{w_0}{r+v_0}\right)^u$$

zu zeigen. Unter Benutzung von (6. 7) und (6. 4) erhält man

$$\frac{1}{r}\sum_u \binom{r}{u}(r+u)\left(\frac{w_0}{r+w_0}\right)^u \leqq \frac{1}{r}\left(\sum_u \frac{r^{u+1}}{u!}\left(\frac{w_0}{r+w_0}\right)^u + \frac{rw_0}{r+w_0}\right)$$
$$\leqq \frac{r}{r+w_0}\sum_u \frac{w_0^u}{u!} + \frac{w_0}{r+w_0} = \frac{r}{r+w_0} + \frac{w_0}{r+w_0} = 1,$$

also die linke Seite von (6. 8). Die rechte Seite von (6. 8) ergibt sich schließlich mit (6. 5), (6. 6), (6. 4) und $v_0 = \frac{1}{2\sqrt{2}}(2 - w_0)$ folgendermaßen:

$$\frac{1}{r}\sum_u \binom{r}{u}(r+u)\left(\frac{w_0}{r+v_0}\right)^u \geqq \sum_u \binom{r}{u}\left(\frac{w_0}{r+v_0}\right)^u + \frac{w_0}{r+v_0}$$
$$\geqq \sum_{u\leq r}\left(1 - \frac{u(u-1)}{2r}\right)\frac{r^u}{u!}\left(\frac{w_0}{r+v_0}\right)^u + \frac{w_0}{r+v_0}$$
$$= \sum_u \frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u - \sum_{u>r}\frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u - \sum_{3\leq u\leq r}\frac{u(u-1)}{2r}\,\frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u + \frac{w_0}{r+v_0}$$
$$\geqq \sum_u \frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u - \frac{1}{r(r+1)}\left(\frac{rw_0}{r+v_0}\right)^2 \sum_{u>r}\frac{1}{(u-2)!}\left(\frac{rw_0}{r+v_0}\right)^{u-2}$$
$$- \frac{1}{2r}\left(\frac{rw_0}{r+v_0}\right)^2 \sum_{3\leq u\leq r}\frac{1}{(u-2)!}\left(\frac{rw_0}{r+v_0}\right)^{u-2} + \frac{w_0}{r+v_0}$$
$$\geqq \sum_u \frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u - \frac{1}{2r}\left(\frac{rw_0}{r+v_0}\right)^2 \sum_u \frac{1}{u!}\left(\frac{rw_0}{r+v_0}\right)^u + \frac{w_0}{r+v_0}$$
$$= \left(1 - \frac{r}{2}\left(\frac{w_0}{r+v_0}\right)^2\right)\sum_u \frac{w_0^u}{u!}\left(\frac{r}{r+v_0}\right)^u + \frac{w_0}{r+v_0}$$
$$\geqq \left(1 - \frac{r}{2}\left(\frac{w_0}{r+v_0}\right)^2\right)\sum_u \frac{w_0^u}{u!}\left(1 - \frac{uv_0}{r+v_0}\right) + \frac{w_0}{r+v_0}$$
$$= \left(1 - \frac{r}{2}\left(\frac{w_0}{r+v_0}\right)^2\right)\left(1 - \frac{w_0 v_0}{r+v_0}\sqrt{2}\right) + \frac{w_0}{r+v_0}$$
$$\geqq 1 - \frac{rw_0^2}{2(r+v_0)^2} - \frac{w_0 v_0\sqrt{2}}{r+v_0} + \frac{w_0}{r+v_0}$$
$$\geqq 1.$$

24. *Beweis von* (4. 7). x_0 ist Lösung von (6. 2). Es gilt also mit $\binom{0}{0} = 1$

$$r = \sum_u \binom{r}{u}(r+u)\,x_0^u = r\left(\sum_u \binom{r}{u} x_0^u + x_0 \sum_g \binom{r-1}{g} x_0^g\right)$$

und weiter

$$1 - \sum_u \binom{r}{u} x_0^u = x_0 \sum_g \binom{r-1}{g} x_0^g,$$

was in (6. 1) eingesetzt

$$F(r) = 2^r x_0^{r+1} \sum_g \binom{r-1}{g} x_0^g \tag{6. 9}$$

ergibt. (6. 9) ist monoton in x_0, so daß man mit Satz 6. 1

$$2^r \left(\frac{w_0}{r+w_0}\right)^{r+1} \sum_g \binom{r-1}{g} \left(\frac{w_0}{r+w_0}\right)^g \leqq F(r) \leqq 2^r \left(\frac{w_0}{r+v_0}\right)^{r+1} \sum_g \binom{r-1}{g} \left(\frac{w_0}{r+v_0}\right)^g$$

erhält. Schätzt man die Summen noch durch

$$\sum_g \binom{r-1}{g} \left(\frac{w_0}{r+w_0}\right)^g \geqq 1$$

und

$$\sum_g \binom{r-1}{g} \left(\frac{w_0}{r+v_0}\right)^g \leqq \sum_g \frac{(r-1)^g}{g!} \left(\frac{w_0}{r+v_0}\right)^g = \mathfrak{Cof}\left(\frac{(r-1)\,w_0}{r+v_0}\right) \leqq \mathfrak{Cof}\, w_0 = \sqrt{2}$$

ab, so folgt (4. 7).

Die folgende Tabelle enthält die numerischen Werte von $F(r)$ und den Schranken in (4. 7) für die ersten r.

r	$\frac{1}{2}\left(\frac{2w_0}{r+w_0}\right)^{r+1}$	$F(r)$	$\frac{1}{\sqrt{2}}\left(\frac{2w_0}{r+v_0}\right)^{r+1}$
1	0,435	0,5	1,14
2	0,114	0,148	0,281
3	0,021	0,0294	0,052
4	0,0030	0,00443	0,0075

25. Es fehlt jetzt noch der Nachweis von (4. 8):

$$F(r) = c\left(\frac{2w_0}{r}\right)^{r+1}\left(1 + O\left(\frac{1}{r}\right)\right), \qquad c = \frac{1}{\sqrt{2}}\, e^{\frac{w_0}{2\sqrt{2}} - 1} = 0{,}3552\ldots.$$

Aus (4. 7) folgt nur, daß $F(r)$ und $\left(\frac{2w_0}{r}\right)^{r+1}$ gleiche Größenordnung haben und daß

$$c_1 \leqq \varliminf_{r\to\infty} \frac{F(r)}{\left(\frac{2w_0}{r}\right)^{r+1}} \leqq \varlimsup_{r\to\infty} \frac{F(r)}{\left(\frac{2w_0}{r}\right)^{r+1}} \leqq c_2$$

mit $c_1 = \frac{1}{2}\, e^{-w_0} = 0{,}2071\ldots$ und $c_2 = \frac{1}{\sqrt{2}}\, e^{-v_0} = 0{,}4761\ldots$ gilt.

Gesucht ist zunächst ein asymptotischer Ausdruck für die Lösung $x_0(r)$ von (6. 2) oder, wenn man den Ansatz $x_0(r) = \frac{1}{r}\, w(r)$ macht, für $w(r)$, und zwar wird die Genauigkeit $1 + O\left(\frac{1}{r^2}\right)$ benötigt. Aus dem bisherigen Überlegungen (Satz 6. 1) erhält man aber nur $w(r) = w_0\left(1 + O\left(\frac{1}{r}\right)\right)$.

Wünschenswert wäre ein Potenzreihenansatz

$$w(r) = w_0 + w_1 \frac{1}{r} + w_2 \frac{1}{r^2} + \cdots. \tag{6.10}$$

Die Möglichkeit und Konvergenz dieses Ansatzes wird mit funktionentheoretischen Mitteln gezeigt.

Setzt man $x = \frac{w}{r}$ in (6. 2) ein, so folgt

$$\sum_u \binom{r}{u} (r+u) \frac{w^u}{r^u} - r = 0 \tag{6.11}$$

und bei Division durch r und Substitution von $z = \frac{1}{r}$

$$f(w, z) = \sum_u (1-z)\left(\frac{1}{2} - z\right) \cdots \left(\frac{1}{u-1} - z\right)\left(\frac{1}{u} + z\right) w^u - 1 = 0. \tag{6.12}$$

$f(w, z)$ ist vorerst nur für reelle w und $z = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ definiert.

Hilfssatz 6. 1. *$f(w, z)$ läßt sich zu einer im Gebiete G: $(1 + |z|)\,|w| < 1$ regulären Funktion der beiden komplexen Variabeln w, z fortsetzen.*

Beweis. Es genügt zu zeigen, daß die Summe in (6. 12) in jedem abgeschlossenen Teilgebiet von G gleichmäßig konvergiert, und dazu ist hinreichend, daß für alle $(1 + |z|)\,|w| \leqq k < 1$ eine konvergente Majorante existiert[19]. Nun ist aber

$$\begin{aligned} &\left|(1-z)\left(\frac{1}{2} - z\right) \cdots \left(\frac{1}{u-1} - z\right)\left(\frac{1}{u} + z\right) w^u\right| \\ &\leqq (1 + |z|)\left(\frac{1}{2} + |z|\right) \cdots \left(\frac{1}{u-1} + |z|\right)\left(\frac{1}{u} + |z|\right) |w|^u \\ &\leqq (1 + |z|)^u\, |w|^u \leqq k^u \end{aligned}$$

und $\sum_u k^u$ konvergiert wegen $k < 1$.

26. Jetzt benutze ich folgenden Satz über implizite analytische Funktionen[20].

Satz. *Ist $f(w, z)$ eine komplexwertige Funktion der beiden komplexen Variabeln w, z, die folgenden Bedingungen genügt:*

1. *Es gibt ein Paar w^*, z^* mit $f(w^*, z^*) = 0$,*
2. *$f(w, z)$ ist in einer Umgebung von w^*, z^* regulär,*
3. $$\frac{\partial}{\partial w} f(w, z^*)\Big|_{w = w^*} \neq 0,$$

so existiert eine Funktion $w = \varphi(z)$ mit $w^* = \varphi(z^*)$, die in einer Umgebung von z^* regulär ist und $f(\varphi(z), z) = 0$ erfüllt.

Die Voraussetzungen 1., 2., 3. treffen für das $f(w, z)$ aus (6. 12) zu. Denn für $w^* = w_0 = ln\,(1 + \sqrt{2})$, $z^* = 0$ ist $(1 + |z^*|)\,|w^*| = w_0 = 0{,}88\ldots < 1$, d. h. $w_0, 0$

[19]) Vgl. Knopp [11], § 19.

[20]) S. z. B. Osgood [14], Kapitel I, § 6.

gehört zum Regularitätsgebiet G von $f(w, z)$ und 2. ist erfüllt. Dann erhalte ich $f(w_0, 0)$ einfach durch Einsetzen in (6. 12):

$$f(w_0, 0) = \sum_u \frac{w_0^u}{u!} - 1 = \mathfrak{Sin}\, w_0 - 1 = 0,$$

so daß 1. erfüllt ist. Schließlich ist

$$\frac{\partial}{\partial w} f(w, 0)\Big|_{w=w_0} = \frac{\partial}{\partial w} (\mathfrak{Sin}\, w - 1)\Big|_{w=w_0} = \mathfrak{Cos}\, w_0 = \sqrt{2} \neq 0.$$

Also gilt auch 3.

Es gibt also eine reguläre Funktion $w = w(z)$, die $f(w, z) = 0$ in einer Umgebung U von $z = 0$ auflöst und sich dort in eine konvergente Potenzreihe mit dem niedrigsten Koeffizienten $w_0 = ln\,(1 + \sqrt{2})$ entwickeln läßt:

$$w(z) = w_0 + w_1 z + w_2 z^2 + \cdots.$$

In U konvergieren dann auch die Reihen

$$w(z)^u = w_0^u + u w_0^{u-1} w_1 z + \cdots$$

und ebenso, da

$$(1 - z)\left(\frac{1}{2} - z\right) \cdots \left(\frac{u-1}{1} - z\right)\left(\frac{1}{u} + z\right) = \frac{1}{u!}\left(1 + \frac{3u - u^2}{2} z + \cdots\right)$$

für alle $z \neq \infty$ regulär ist, die Reihen

$$(1 - z)\left(\frac{1}{2} - z\right) \cdots \left(\frac{1}{u-1} - z\right)\left(\frac{1}{u} + z\right) w(z)^u$$
$$= \frac{1}{u!}\left(w_0^u + \left(w_0^u \frac{3u - u^2}{2} + u w_0^{u-1} w_1\right) z + \cdots\right).$$

Setze ich das in (6. 12) ein:

$$(6.\ 13)\quad f(w, z) = \sum_u \frac{1}{u!}\left(w_0^u + \left(w_0^u \frac{3u - u^2}{2} + u w_0^{u-1} w_1\right) z + \cdots\right) - 1$$

und schränke das Gebiet U so zu V ein, daß $(1 + |z|)\,|w(z)| \leqq k_0 < 1$ gilt für alle Paare $w(z), z$ in V, so konvergiert (6. 13) in V gleichmäßig. Nach dem Weierstraßschen Doppelreihensatz [21]) erhält man also die Potenzreihe für $f(w(z), z)$ durch Umordnen:

$$(6.\ 14)\quad f(w(z), z) = \left(\sum_u \frac{w_0}{u_u!} - 1\right) + \left(\sum_u \frac{1}{u!}\left(w_0^u \frac{3u - u^2}{2} + u w_0^{u-1} w_1\right)\right) z + \cdots$$

Aus $f(w(z), z) = 0$ folgt jetzt durch Koeffizientenvergleich

(a) $$\sum_u \frac{w_0^u}{u!} - 1 = 0, \qquad \text{(durch } w_0 \text{ erfüllt.)}$$

(b) $$\sum_u \frac{1}{u!}\left(w_0^u \frac{3u - u^2}{2} + u w_0^{u-1} w_1\right) = 0,$$

.

.

.

[21]) Vgl. Knopp [11], § 20.

Aus (b) wird w_1 bestimmt. Es ist

$$w_1\sqrt{2} = w_1 \operatorname{Cof} w_0 = w_1 \sum_u \frac{w_0^{u-1}}{(u-1)!}$$

$$= -\sum_u \frac{1}{u!} w_0^u \frac{3u-u^2}{2} \text{ (nach (b))}$$

$$= -\left(w_0 + \sum_{u\geq 3} \frac{w_0^u}{(u-1)!} - \frac{1}{2}\sum_{u\geq 3}\frac{w_0^u}{(u-2)!}\right)$$

$$= -\left(w_0 \operatorname{Cof} w_0 - \frac{w_0^2}{2} \operatorname{Sin} w_0\right) = \frac{w_0^2}{2} - w_0\sqrt{2}.$$

Hieraus folgt

$$w_1 = w_0\left(\frac{w_0}{2\sqrt{2}} - 1\right) \tag{6.15}$$

und

$$w(z) = w_0 + w_0\left(\frac{w_0}{2\sqrt{2}} - 1\right) z + \cdots. \tag{6.16}$$

Damit ist $w(z)$ und also auch $x_0(r) = \frac{1}{r} w\left(\frac{1}{r}\right)$ hinreichend genau bestimmt.

Beweis von (4. 7). Der Beweis erfolgt in drei Schritten.

I. Man berechnet zunächst die Summe in (6. 9):

$$S = \sum_g \binom{r-1}{g} x_0^g = \sum_g \frac{1}{g!}(1-z)\cdots(1-gz)\, w(z)^g$$

konvergiert ebenso wie die Reihe für $f(w, z)$ in G gleichmäßig (Beweis wie zu Hilfssatz 6. 1), so daß in V

$$S = \sum_g \frac{1}{g!}\left(1 - \frac{g(g-1)}{2} z + \cdots\right)(w_0^g + g w_0^{g-1} w_1 z + \cdots)$$

$$= \sum_g \frac{1}{g!}(w_0^g + c_{g1} z + \cdots)$$

$$= \left(\sum_g \frac{w_0^g}{g!}\right) + d_1 z + \cdots$$

$$= \operatorname{Cof} w_0 + O(z) = \sqrt{2} + O(z)$$

und somit

$$\sum_g \binom{r-1}{g} x_0^g = \sqrt{2}\left(1 + O\left(\frac{1}{r}\right)\right) \tag{6.17}$$

gilt.

II. Zur Bestimmung von x_0^{r+1} aus (6. 9) brauche ich

Hilfssatz 6. 2. *Ist* $1 + c_1 z + c_2 z^2 + \cdots$ *in einer Umgebung von* $z = 0$ *konvergent, so gilt*

$$(1 + c_1 z + \cdots)^{\frac{1}{z}} = e^{c_1}(1 + O(z)). \tag{6.18}$$

Beweis. In einer Umgebung von $z = 0$ konvergiert obige Reihe und es gilt $|c_1 z + \cdots| \leqq a < 1$. Dort ist dann (alle Reihen sind gleichmäßig konvergent)

$$ln\,(1 + c_1 z + \cdots)^{\frac{1}{z}} = \frac{1}{z} \sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+1}}{\nu} (c_1 z + \cdots)^{\nu}$$

$$= \frac{1}{z}(c_1 z + d_2 z^2 + \cdots) = c_1 + d_2 z + \cdots$$

und weiter

$$(1 + c_1 z + \cdots)^{\frac{1}{z}} = e^{c_1 + d_2 z + \cdots}$$

$$= e^{c_1} \sum_{\nu=0}^{\infty} \frac{(d_2 z + \cdots)^{\nu}}{\nu!} = e^{c_1}(1 + d_2 z + \cdots)$$

$$= e^{c_1}(1 + O(z)).$$

In einer Umgebung von $z = 0$ ist also nach (6. 16) und (6. 18)

$$w(z)^{r+1} = w(z)\, w(z)^{\frac{1}{z}}$$

$$= w_0^{r+1}(1 + O(z))\left(1 + \left(\frac{w_0}{2\sqrt{2}} - 1\right) z + \cdots\right)^{\frac{1}{z}}$$

$$= w_0^{r+1}\, e^{\frac{w_0}{2\sqrt{2}} - 1}\,(1 + O(z))$$

und daher

$$x_0(r)^{r+1} = \left(\frac{w_0}{r}\right)^{r+1} e^{\frac{w_0}{2\sqrt{2}} - 1}\left(1 + O\left(\frac{1}{r}\right)\right). \tag{6. 19}$$

III. Wenn man (6. 17) und (6. 19) in (6. 9) einsetzt, folgt das Endresultat

$$F(r) = \frac{1}{\sqrt{2}}\, e^{\frac{w_0}{2\sqrt{2}} - 1}\left(\frac{2 w_0}{r}\right)^{r+1}\left(1 + O\left(\frac{1}{r}\right)\right) \tag{6. 20}$$

und damit ist Satz 4. 2 vollständig bewiesen.

Literaturverzeichnis.

[1] *L. Chatrovsky*, Sur le théorème de Erdös-Raikov. Izvestiya Akad. Nauk SSSR **9** (1945), 301–310.

[2] *L. Cheo*, On the density of sets of Gaussian integers. Amer. Math. Monthly **58** (1951), 618–620.

[3] *P. Erdös*, On the arithmetical density of the sum of two sequences one of which forms a basis for the integers Acta arith. Warszawa **1** (1936), 197–200.

[4] *P. Halmos*, Measure Theory, New York 1950.

[5] *F. Kasch*, Abschätzung der Dichte von Summenmengen I. Math. Zeitschr. **62** (1955), 368–387.

[6] *F. Kasch*, Abschätzung der Dichte von Summenmengen II. Math. Zeitschr. **64** (1956), 243–257.

[7] *F. Kasch*, Wesentliche Komponenten bei Gitterpunktmengen I. Crelles Journal **197** (1957), 208–215.

[8] *F. Kasch*, Wesentliche Komponenten bei Gitterpunktmengen II. Crelles Journal **199** (1958), 53–55.

[9] *H. Klöter* u. *A. Stöhr*, Wesentliche Komponenten und asymptotische Dichte. Crelles Journal **194** (1955), 210–217.

[10] *M. Kneser*, Abschätzungen der asymptotischen Dichte von Summenmengen. Math. Zeitschr. **58** (1953), 459–484.

[11] *K. Knopp*, Funktionentheorie I. Sammlung Göschen, 1955.

[12] *E. Landau*, Über einige neuere Fortschritte der additiven Zahlentheorie. Cambr. Tracts. No. **35** (1937).

[13] *H. B. Mann*, A proof of the fundamental theorem on the density of sums of sets of positive integers. Ann. of Math. (2) **43** (1942), 523—527.
[14] *W. F. Osgood*, Funktionentheorie II, Leipzig, Berlin 1929.
[15] *H. Ostmann*, Additive Zahlentheorie I, Berlin, Göttingen, Heidelberg 1956.
[16] *D. Raikov*, On the addition of pointsets in the sense of Schnirelmann. Rec. math. Moscou (2) **5** (1939), 425—440.
[17] *H. Rohrbach*, Einige neuere Untersuchungen über die Dichte in der additiven Zahlentheorie. Jahresber. d. dtschen Math. Vereingg. **48** (1939), 199—236.
[18] *H. Schell*, Über einige Probleme der Addition von linearen Punktmengen. Crelles Journal **200** (1958), 52—88.
[19] *L. Schnirelmann*, Über additive Eigenschaften von Zahlen. Math. Ann. **107** (1933), 649—690.
[20] *L. Schnirelmann*, On the addition of sequences and sets. Rec. math. Moscou (2) **5** (1939), 211—215.
[21] *A. Stöhr*, Bemerkungen zur additiven Zahlentheorie I. Mittlere Ordnung. Crelles Journal **183** (1941), 168—174.

Eingegangen 18. November 1958.

Über die explizite Automorphismengruppe von speziellen Graphen[1]).

Von *Herbert Izbicki* in Wien.

Einleitung.

Ein *Automorphismus* eines endlichen Graphen ist eine Permutation seiner Knotenpunkte, die folgende Bedingung erfüllt: Es seien p und q zwei beliebige (nicht notwendigerweise verschiedene) Knotenpunkte des Graphen, und es werde p durch die Permutation in p' und q in q' übergeführt; dann sind die zwei Knotenpunkte p und q durch die gleiche Anzahl von Kanten verbunden wie die zwei Knotenpunkte p' und q'. Sämtliche Automorphismen des endlichen Graphen X bilden eine Permutationsgruppe, die in dieser Arbeit die *explizite Automorphismengruppe des Graphen* X genannt und mit $G(X)$ bezeichnet wird. Eine mit dieser Gruppe isomorphe abstrakte Gruppe nennt man *Automorphismengruppe des Graphen* X.

Graphen ohne Schlingen, in denen zwei Knotenpunkte durch höchstens eine Kante verbunden sind, heißen *spezielle* Graphen. Zweck dieser Arbeit ist, Zusammenhänge zwischen der Struktur endlicher spezieller Graphen und ihren expliziten Automorphismengruppen aufzuzeigen. Dieses Thema wurde von I. N. Kagno [1] behandelt, der sich hauptsächlich mit speziellen Graphen einer Ordnung (Anzahl der Knotenpunkte) $\leqq 6$ und mit Permutationsgruppen vom Grad (Anzahl der Elemente) $\leqq 6$ befaßte. Die vorliegende Arbeit ist davon unabhängig. Sie befaßt sich auch nicht mit der Konstruktion von speziellen Graphen mit zu einer gegebenen endlichen abstrakten Gruppe isomorphen Automorphismengruppen.

Die V-Darstellung.

Unter einem *vollständigen Graphen* versteht man einen Graphen ohne Schlingen, in dem je zwei verschiedene Knotenpunkte durch genau eine Kante verbunden sind. Ein Knotenpunkt allein ist auch ein vollständiger Graph.

Es sei ein endlicher spezieller Graph K folgendermaßen konstruiert:

$K_1, K_2, \ldots, K_r$ $(r \geqq 1)$ seien endliche, paarweise fremde, nicht leere vollständige Graphen. K' sei ein spezieller Graph mit den Knotenpunkten $\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_r$. K' hat mit $K_1, K_2, \ldots, K_r$ keinen Knotenpunkt gemeinsam. Falls $\mathfrak{k}_i$ und $\mathfrak{k}_j$ $(1 \leqq i \neq j \leqq r)$ in K' mit einer Kante verbunden sind, sind in K alle Knotenpunkte von K_i mit allen Knotenpunkten von K_j durch Kanten verbunden; falls $\mathfrak{k}_i$ und $\mathfrak{k}_j$ in K' nicht mit einer Kante verbunden sind, ist in K kein Knotenpunkt von K_i mit einem Knotenpunkt von K_j verbunden. Offenbar ist der spezielle Graph K durch seine *Vollständigkeitskomplexe* $K_1, K_2, \ldots, K_r$ und durch K' eindeutig festgelegt. Derjenige Graph K^*, welcher aus K'

[1]) Der Autor möchte Herrn Dr. *G. A. Dirac* für die bei der Abfassung dieser Arbeit geleistete Hilfe Dank sagen.

entsteht, indem man für $i = 1, 2, \ldots, r$ auf $\mathfrak{k}_i$ so viele Schlingen konstruiert, wie es in K_i Knotenpunkte gibt, wird der *Grundgraph von K* genannt. Offenbar ist K durch K^* eindeutig bestimmt. Ein Beispiel ist in Abbildung 1 gezeigt. Die Knotenpunkte von K_i sind

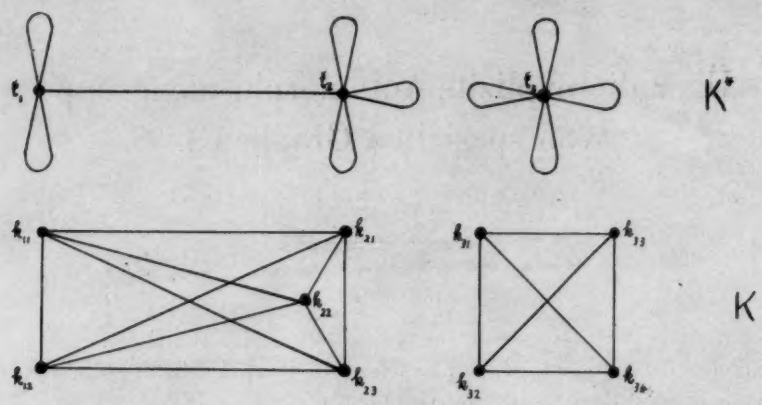


Abbildung 1

mit $k_{i1}, k_{i2}, \ldots, k_{ik_i}$ bezeichnet. Einen Graphen von der Gestalt von K nennen wir einen *Graphen in V-Darstellung*, (als Abkürzung von Graph in Vollständigkeitskomplexdarstellung).

Jeder endliche spezielle Graph X läßt sich als Graph in V-Darstellung konstruieren. Eine V-Darstellung des Graphen ist z. B. dadurch gegeben, daß man jeden Knotenpunkt als vollständigen Graphen auffaßt. Anders ausgedrückt: als K^* nimmt man X.

Ein Graph X liegt in *minimaler V-Darstellung* vor, wenn er (I) in V-Darstellung gegeben ist, und wenn es (II) für X keine V-Darstellung mit einer geringeren Anzahl von Vollständigkeitskomplexen gibt. Klarerweise hat jeder endliche spezielle Graph mindestens eine minimale V-Darstellung (da ja jeder solche Graph mindestens eine V-Darstellung hat).

Die explizite Automorphismengruppe eines in minimaler V-Darstellung gegebenen Graphen.

Es sei K ein endlicher spezieller Graph in minimaler V-Darstellung mit den Vollständigkeitskomplexen $K_1, K_2, \ldots, K_r$ $(r \geqq 1)$ und dem Grundgraphen K^*, wobei die Knotenpunkte von K_i, in den Anzahlen k_i, mit $k_{i1}, k_{i2}, \ldots, k_{ik_i}$ bezeichnet sind $(1 \leqq i \leqq r)$ und die Knotenpunkte von K^* mit $\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_r$.

Es sei $\mathfrak{P}\{k_{i1}, \ldots, k_{ik_i}\}$ oder $\mathfrak{P}\{K_i\}$ die Gruppe aller Permutationen der Knotenpunkte von K_i. Es sei $G(K^*)$ die explizite Automorphismengruppe von K^*. Es sei $\mathfrak{R}\{k_{11}, \ldots, k_{1k_1}, k_{21}, \ldots, k_{2k_2}, \ldots, k_{r1}, \ldots, k_{rk_r}\}$ oder $\mathfrak{R}\{K\}$ die Permutationsgruppe der Knotenpunkte von K, die man aus $G(K^*)$ gewinnt, indem man die Permutation

$$\begin{pmatrix} \mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_r \\ \mathfrak{k}_i, \mathfrak{k}_j, \ldots, \mathfrak{k}_t \end{pmatrix}$$

aus $G(K^*)$ durch die Permutation

$$\begin{pmatrix} k_{11}, k_{12}, \ldots, k_{1k_1}, k_{21}, k_{22}, \ldots, k_{2k_2}, \ldots, k_{r1}, k_{r2}, \ldots, k_{rk_r} \\ k_{i1}, k_{i2}, \ldots, k_{ik_i}, k_{j1}, k_{j2}, \ldots, k_{jk_j}, \ldots, k_{t1}, k_{t2}, \ldots, k_{tk_t} \end{pmatrix}$$

der Knotenpunkte von K ersetzt. Das ist sinnvoll, weil, falls die Permutation

$$\begin{pmatrix} \mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_r \\ \mathfrak{k}_i, \mathfrak{k}_j, \ldots, \mathfrak{k}_t \end{pmatrix}$$

zu $G(K^*)$ gehört, entsprechende Knotenpunkte in dieser Permutation, also $\mathfrak{k}_1$ und $\mathfrak{k}_i$, $\mathfrak{k}_2$ und $\mathfrak{k}_j, \ldots, \mathfrak{k}_r$ und $\mathfrak{k}_t$, dieselbe Anzahl von Schlingen haben und daher K_1 und K_i, K_2 und $K_j, \ldots, K_r$ und K_t dieselbe Anzahl von Knotenpunkten haben, so daß $k_1 = k_i$, $k_2 = k_j, \ldots, k_r = k_t$ ist.

Wir beweisen nunmehr den folgenden

Hauptsatz. *Die explizite Automorphismengruppe $G(K)$ von K ist, mit den obigen Bezeichnungen*

$$\mathfrak{R}\{K\} \prod_{i=1}^{r} \mathfrak{P}\{K_i\}.$$

Dieser Satz zeigt, wie man die explizite Automorphismengruppe eines endlichen speziellen Graphen durch die Darstellung des Graphen in minimaler V-Darstellung bekommt.

Beweis des Hauptsatzes:

Satz 1. *Sämtliche Knotenpunkte von $K_i (1 \leqq i \leqq r)$ sind mit derselben Knotenpunktmenge außerhalb K_i verbunden.*

Das folgt sofort aus der Definition der V-Darstellung.

Satz 2.

$$\mathfrak{R}\{K\} \prod_{i=1}^{r} \mathfrak{P}\{K_i\} \leqq G(K).$$

Beweis: Nach der Definition der V-Darstellung liefert ein Automorphismus des Grundgraphen K^* einen Automorphismus von K, wenn die Permutationen der Knotenpunkte von K^* und K wie oben zusammenhängen. Ferner folgt aus der Definition der V-Darstellung und aus Satz 1 sofort, daß die Vertauschung zweier beliebiger Knotenpunkte aus ein- und demselben Vollständigkeitskomplex einen Automorphismus des Graphen ergibt. Damit ergibt sich Satz 2.

In dem übrigbleibenden Teil des Beweises des Hauptsatzes ist noch zu zeigen, daß K keine anderen Automorphismen hat als die auf der linken Seite der obigen Ungleichung gezählten.

Satz 3. *Es seien x, y irgend zwei Knotenpunkte von K, die zu zwei verschiedenen der Vollständigkeitskomplexe $K_1, K_2, \ldots, K_r$ gehören und die in K durch eine Kante verbunden sind. Dann gehört die Permutation (x, y) nicht zu $G(K)$.*

Beweis: Nehmen wir an, es sei $x \in K_1$, $y \in K_2$ und $(x, y) \in G(K)$. Dann sind nach der Definition der V-Darstellung sämtliche Knotenpunkte von K_1 mit sämtlichen Knotenpunkten von K_2 verbunden, und K_1, K_2 bilden zusammen einen vollständigen Graphen mit $k_1 + k_2$ Knotenpunkten. Da $(x, y) \in G(K)$ ist, sind x und y mit derselben Menge von Knotenpunkten $\neq x$, $\neq y$ verbunden. Aus Satz 1 folgt, daß sämtliche Knotenpunkte aus $K_1 + K_2$ mit derselben Menge von Knotenpunkten aus $K_3, \ldots, K_r$ verbunden sind.

Wegen dieser Tatsachen bekommt man durch Zusammenschließen von K_1 und K_2 zu einem einzigen Vollständigkeitskomplex eine neue V-Darstellung. Diese hat aber weniger Vollständigkeitskomplexe, was ein Widerspruch ist, da ja die V-Darstellung von K minimal ist. Damit ist Satz 3 bewiesen.

Nun sei L ein zu K isomorpher Graph, wobei K und L keinen Knotenpunkt gemeinsam haben. Es gibt mindestens eine minimale V-Darstellung von L, und jede solche hat dieselbe Anzahl von Vollständigkeitskomplexen wie K, weil K und L isomorph sind. Es sei eine minimale V-Darstellung von L vorhanden mit den Vollständigkeitskomplexen $L_1, L_2, \ldots, L_r$; die Knotenpunkte von L_i seien $l_{i1}, l_{i2}, \ldots, l_{iu_i}$, und der Grundgraph L^* habe die Knotenpunkte $\mathfrak{l}_1, \mathfrak{l}_2, \ldots, \mathfrak{l}_r$, wobei sich jeweils $\mathfrak{l}_i$ und L_i entsprechen.

Es sei I ein beliebiger Isomorphismus zwischen K und L.

Nehmen wir an, k_{11} entspreche l_{11} in I. Dann haben K_1 und L_1 dieselbe Zahl von Knotenpunkten, und die Knotenpunkte von K_1 werden durch I auf die Knotenpunkte von L_1 abgebildet. Beweis: Wenn dies nicht wahr ist, können wir o. *B.* d. A. annehmen, daß sich k_{12} in I mit $l_{21} \in L_2$ entspricht. l_{11} und l_{21} sind dann durch eine Kante verbunden, weil k_{11} und k_{12} es sind. Nach Satz 2 ist die Permutation $(k_{11}, k_{12}) \in G(K)$, und daher ist $(l_{11}, l_{21}) \in G(L)$. Es ist aber $l_{11} \in L_1$ und $l_{21} \in L_2$, so daß ein Widerspruch zu Satz 3 vorliegt. Damit ist die Behauptung bewiesen.

Nunmehr folgt, daß — bei geeigneter Bezeichnung der Vollständigkeitskomplexe (geeigneter Auswahl der Indizes $1, 2, \ldots, r$) in dieser minimalen V-Darstellung von L — der Isomorphismus I für alle $i (1 \leqq i \leqq r)$ die Knotenpunkte von K_i genau auf die Knotenpunkte von L_i abbildet. Die Abbildung $\mathfrak{k}_i \longleftrightarrow \mathfrak{l}_i$ ergibt daher einen Isomorphismus von K^* und L^*.

Aus diesen Tatsachen läßt sich sofort schließen:

Satz 4. *Die minimale V-Darstellung eines Graphen ist, wenn man von Umbenennungen absieht, eindeutig bestimmt.*

Satz 5. *Jeder Automorphismus des Graphen K stellt sich als Produkt einer Permutation aus $\prod_{i=1}^{r} \mathfrak{P}\{K_i\}$ und einer aus $\mathfrak{R}\{K\}$ dar.*

Aus Satz 2 und Satz 5 ergibt sich der Hauptsatz:

$$G(K) = \mathfrak{R}(K) \prod_{i=1}^{r} \mathfrak{P}\{K_i\}.$$

Anwendung des Hauptsatzes.

Der Hauptsatz kann offenbar bei der Berechnung der expliziten Automorphismengruppe eines Graphen nützlich sein.

Ist umgekehrt eine Gruppe in der Gestalt

$$\mathfrak{G} = \mathfrak{R} \prod_{i=1}^{r} \mathfrak{P}_i$$

vorgegeben, wobei $\mathfrak{P}_1, \mathfrak{P}_2 \ldots, \mathfrak{P}_r$ paarweise elementfremde symmetrische Permutationsgruppen sind und $\mathfrak{R}$ eine endliche Gruppe ist, die mit $\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_r$ keine gemeinsamen Elemente hat, und sucht man einen Graphen X mit $G(X) \cong \mathfrak{G}$, so kann man folgenden Weg beschreiten: Man bestimme einen Graphen K^* mit den Knotenpunkten $\mathfrak{k}_1, \mathfrak{k}_2, \ldots, \mathfrak{k}_r$, in dem $\mathfrak{k}_i$ so viele Schlingen hat wie $\mathfrak{P}_i$ Elemente, und der eine mit $\mathfrak{R}$ isomorphe Automorphismengruppe hat. Hat man einen solchen gefunden, so gehe man von K^* zu K über; wenn dann die resultierende V-Darstellung minimal ist, so ist $G(K) \cong \mathfrak{G}$.

Aus dem Hauptsatz folgt, daß eine Permutationsgruppe nur dann die explizite Automorphismengruppe eines endlichen speziellen Graphen ist, wenn sie in der Form

$$\mathfrak{R}(K) \prod_{i=1}^{r} \mathfrak{P}(K_i)$$

dargestellt werden kann. Nicht alle Permutationsgruppen haben diese Eigenschaft. (Für Beispiele siehe [1].)

Literatur.

[1] *I. N. Kagno*, Linear graphs of Degree $\leqq 6$ and their groups, Amer. J. of Math. 68 (1946), 505–520.

Eingegangen 27. Dezember 1958.

Die Hauptgruppen singulärer konformeuklidischer Räume.

Karl Dörge zum 60. Geburtstag.

Von *M. Pinl* in Köln.

Die euklidischen Bewegungen der Ebene E_2 haben bekanntlich die Eigenschaft, das Paar M_0^2 der absoluten Kreispunkte

$$x_0 = 0,\ x_1^2 + x_2^2 = 0 \tag{1}$$

auf der Ferngeraden $x_0 = 0$ der auf (rechtwinklige) homogene Koordinaten $x_0 : x_1 : x$, bezogenen E_2 in sich selbst überzuführen. Die allgemeinsten ternären Kollineationen welchen diese Eigenschaft zukommt, sind jedoch die Ähnlichkeitstransformationen welche den Winkel zweier Geraden ungeändert lassen, aber die Entfernung zweier Punkte mit einer für alle Punktepaare gleichen Konstanten multiplizieren. Sind $u_0 : u_1 : u_2$ die $x_0 : x_1 : x_2$ entsprechenden Geradenkoordinaten, so kann man die absoluten Kreispunkte (1) als ausgeartete Klassenkurve

$$u_1^2 + u_2^2 = 0 \tag{2}$$

schreiben, deren Gleichung (2) etwa durch den Grenzübergang $\lambda \to 0$ aus

$$Q(u, \lambda) \equiv \lambda^2 u_0^2 + u_1^2 + u_2^2 = 0$$

hervorgeht. Diesen Sachverhalt hat H. Beck gelegentlich folgendermaßen ausgedrückt[1]): „Die euklidische Geometrie ($\lambda = 0$) besteht aus den Trümmern einer elliptischen Geometrie ($\lambda = 1$).“ Doch hat sich bekanntlich ein Rest der elliptischen Geometrie auf der Ferngeraden $x_0 = 0$ erhalten, denn auf dieser ist M_0^2 nicht ausgeartet. Wir nennen daher im folgenden einen n-dimensionalen euklidischen Raum E_n *regulär*, solange seine absolute M_{n-2}^2

$$x_0 = 0,\ x_1^2 + x_2^2 + \cdots x_n^2 = 0,\ n \geqq 2 \tag{3}$$

auf der Fernhyperebene $x_0 = 0$ des auf (rechtwinklige) homogene Koordinaten

$$x_0 : x_1 : \cdots : x_{n-1} : x_n$$

bezogenen E_n nicht ausartet; die Geometrie auf $x_0 = 0$ ist dann wieder eine elliptische Geometrie. Erst wenn wir Ausartungen der absoluten M_{n-2}^2 auf $x_0 = 0$ zulassen, welche deren kanonische Darstellung (3) etwa auf die kanonische Darstellung

$$x_0 = 0,\ x_0^2 + \cdots + x_{n-r}^2 = 0,\ n \geqq 2,\ r = 1, \ldots, n \tag{4}$$

reduzieren, wird für die entsprechenden E_n eine neue Terminologie notwendig. Wir sprechen dann von *r-fach singulären euklidischen* E_n und schreiben für diese $E_n^{(r)}$. Die Zahl r nennen wir den *Defekt* des euklidischen Raumes; die regulären euklidischen Räume $E_n^{(0)}$ sind vom Defekt $r = 0$, die *totalsingulären euklidischen Räume* $E_n^{(n)}$ vom Defekt $r = n$.

Alle singulären $E_n^{(r)}$ lassen sich in reguläre E_N von hinreichend großer Dimensionszahl N einbetten (vgl. § 3). Im E_N ist dann jeder $E_n^{(r)}$ durch eine besondere Lage hin-

[1]) Vgl. *H. Beck*, Koordinatengeometrie, Berlin 1919, S. 139.

sichtlich der absoluten M^2_{N-2} des E_N ausgezeichnet. Aus diesem Grunde hat sich für die *eingebetteten* $E_n^{(r)}$ die Terminologie *r-fach isotrope* E_n eingebürgert (ἴσος τρόπος gleiche Wendung). Die eingebetteten $E_n^{(n)}$ heißen dann auch *totalisotrope*, *vollisotrope*, *ametrische* $E_n^{(n)}$ [2]), ältere Bezeichnungen lauten „Minimalgeraden", „Minimalebenen", „Minimalräume", ...[3]). Erscheint so die euklidische Geometrie nach den vorstehenden (bekannten) Bemerkungen durch den Verlust des Dualitätsprinzips (in den Trümmern) der elliptischen Geometrie belastet, so gewinnt die Automorphismengruppe ihrer absoluten M^2_{n-2} zur Entschädigung einen Parameter (und die euklidische Geometrie damit ihre Ähnlichkeitssätze). Wir bezeichnen diese Automorphismengruppe mit $G_{\frac{n(n+1)}{2}+1}$ und betrachten sie als $\left(\frac{n(n+1)}{2}+1\right)$-parametrige *Kleinsche Hauptgruppe* des E_n, den wir jetzt mit Rücksicht auf das verschiedene Verhalten der Winkel zweier Geraden und der Entfernung zweier Punkte in E_n bei Transformationen dieser Gruppe als n-dimensionalen *konformeuklidischen* E_n bezeichnen.

In § 1 stellen wir uns die Aufgabe, die Parameterzahl der Hauptgruppe eines r-fach singulären konformeuklidischen $E_n^{(r)}$ zu berechnen, und tabellieren unsere Ergebnisse für $n = 1, 2, 3, 4, 5, 6, 7, 8$ und $r = 0, 1, 2, 3, 4, 5, 6, 7, 8$ in § 2. Dabei entspricht der Defekt $r = 0$ dem regulären Fall. Der Defekt $r = n$ der totalsingulären konformeuklidischen $E_n^{(n)}$ führt nach (3) auf

$$x_0 = 0,\ 0 \cdot x_1^2 + 0 \cdot x_2^2 + \cdots + 0 \cdot x_n^2 \equiv 0 \quad \{x_1, \ldots, x_n\},$$

d. h. auf die einzige Forderung $x_0 = 0$ und damit zum affinen n-dimensionalen Raum zurück. Auf diesen Sachverhalt hat in der Theorie der totalisotropen, d. h. der eingebetteten $E_n^{(n)}$ in $E_N^{(0)}$ bereits Herr K. Strubecker hingewiesen und z. B. die Geometrie auf den $E_2^{(2)}$ des konformeuklidischen $E_4^{(0)}$ auf die ebene affine Geometrie zurückgeführt [4]). Auch der Defekt $r = n - 1$ ist von besonderem Interesse, sofern die Bedingungen (3) sich in diesem Falle auf *zwei lineare Bedingungen*

$$x_0 = 0,\ x_1 = 0$$

reduzieren. Variiert man nämlich H. Becks Ausspruch über die euklidische Geometrie auf einen analogen über die affine Geometrie, so besteht die (reguläre) affine Geometrie „aus den Trümmern einer projektiven Geometrie" und die affininvariante Fernhyperebene $x_0 = 0$ bewahrt den letzten Rest der projektiven Geometrie wie in H. Becks Formulierung den der elliptischen Geometrie. Die zusätzliche lineare Forderung $x_1 = 0$ beraubt jetzt auch die Fernhyperebene $x_0 = 0$ ihres allgemein projektiven Charakters und verwandelt den *regulären* n-dimensionalen affinen Raum in einen *einfach singulären* n-dimensionalen *biaffinen Raum*, dessen Hauptgruppe wir die *biaffine* Hauptgruppe nennen. Im Falle gerader Dimensionen $n = 2m$ kann der Defekt r auch den Wert $r = m$ annehmen. Wir sprechen dann von *halbaffinen* Räumen und *halbaffinen* Hauptgruppen. In § 3 behandeln wir Einbettungsfragen und durch Einbettungen induzierte Hauptgruppen.

---

[2]) Vgl. z. B. *F. Klein*, Vorlesungen über höhere Geometrie, Berlin 1926, S. 185. *J. Lense*, Über ametrische Mannigfaltigkeiten und quadratische Differentialformen mit verschwindender Diskriminante, Jahresbericht D. M. V. **35** (1926), 280–294; *M. Pinl*, Quasimetrik auf totalisotropen Flächen. I., Proc. Amsterdam **35** (1932), 1181–1188.

[3]) Vgl. z. B. *Sophus Lie*, Gesammelte Abhandlungen Band II, 1. Teil, S. 128, Leipzig 1935.

[4]) Vgl. *K. Strubecker*, Differentialgeometrie isotroper Mannigfaltigkeiten, Schriftenreihe des Instituts für Mathematik bei der Deutschen Akademie der Wissenschaften zu Berlin, 1, (1957), 143–155.

§ 1. Die Gruppenfolge $G_{\frac{n(n+1)+r(r+1)+2}{2}}$; $r = 0, 1, 2, \ldots, n-1$.

Die absoluten M^2_{n-2} eines r-fach singulären $E^{(r)}_n$ fester Dimension $n \geqq 2$ ist in homogenen Koordinaten $x_0 : x_1 : \cdots : x_{n-1} : x_n$ durch die Gleichungen (1) bzw. (3) bzw. (4) festgelegt. Die gesuchte Hauptgruppe des $E^{(r)}_n$ ist durch die Gesamtheit der automorphen Kollineationen

$$(5) \qquad \varrho x_i = \sum_{k=0}^{n} a_{ik} x'_k, \quad \varrho \neq 0, \quad |a_{ik}| \neq 0, \quad i = 0, 1, 2, \ldots, n$$

der jeweils zugehörigen absoluten M^2_{n-2} gegeben. Für die $(n+1)^2$ komplexen[5]) Parameter a_{ik} der Kollineationen (5) bestehen zunächst unabhängig von r die n Bedingungen

$$(6) \qquad a_{01} = a_{02} = \cdots = a_{0n} = 0,$$

welche den affinen Charakter der Automorphismen sichern (wir sind auf der Fernhyperebene des $E^{(r)}_n$ und bleiben auf dieser). Wegen der Homogenität der Koordinaten und Parameter entfällt außer den n Parametern (6) für die Zählung der wesentlichen Gruppenparameter noch ein weiterer solcher. Es verbleiben (noch unbeeinflußt vom speziellen Wert r) somit $(n+1)^2 - n - 1 = n(n+1)$. Um die Gleichung $x_1^2 + x_2^2 + \cdots + x_{n-r}^2 = 0$ durch $x_1'^2 + x_2'^2 + \cdots + x_{n-r}'^2 = 0$ zu reproduzieren, müssen zunächst die Koeffizienten der $x_1'^2, \ldots, x_{n-r}'^2$, die man aus (5) errechnet, übereinstimmen. Das erfordert $n-r-1$ weitere Bedingungen. Die folgenden r weiteren Quadrate $x_{n-r+1}'^2, \ldots, x_n'^2$, die sich bei der Berechnung aus (5) ergeben, dürfen nicht auftreten; ihre Koeffizienten müssen daher verschwinden. Das sind weitere r Bedingungen. Damit ergeben sich $n(n+1)-(n-r-1)-r = n^2+1$ Parameter (immer noch unabhängig vom speziellen Wert r). Auch Produkte $x'_i x'_k$ mit $i \neq k$ und $i, k = 1, \ldots, n-r, n-r+1, \ldots, n$ dürfen nicht auftreten. Um über die Anzahl der so noch entstehenden restlichen Bedingungen für die Parameter a_{ik} einen Überblick zu gewinnen, setzen wir zur Abkürzung

$$\mathfrak{a}_k = \{a_{k1}, \ldots, a_{k,n-r}\}, \quad k = 1, 2, \ldots, n; \quad r = 0, 1, \ldots, n-1.$$

Nun sei $r = 0$. Dann handelt es sich um n Vektoren $\mathfrak{a}_k$ von je n Komponenten. Dem Verschwinden der Koeffizienten von $x'_i x'_k$, $i, k = 1, \ldots, n$ entsprechen

$n-1$ lineare homogene Orthogonalitätsbedingungen für $\mathfrak{a}_1$ und n Komponenten a_{1k},
$n-2$ lineare homogene Orthogonalitätsbedingungen für $\mathfrak{a}_2$ und n Komponenten a_{2k},

. . . . . .

1 lineare homogene Orthogonalitätsbedingung für $\mathfrak{a}_{n-1}$ und n Komponenten $a_{n-1,k}$,

insgesamt:

$$1 + 2 + \cdots + (n-2) + (n-1) = \frac{n(n-1)}{2} = \frac{n(n-1)}{2} - \frac{0(0+1)}{2}$$

Bedingungen. Daher ist die Parameterzahl s der Hauptgruppe der regulären konformeuklidischen $E^{(0)}_n$ mit

$$s = n^2 + 1 - \frac{n(n-1)}{2} = \frac{n(n+1)}{2} + 1$$

bestätigt.

[5]) Alle Variablen und alle allgemeinen Koeffizienten werden stets als komplexe Zahlen vorausgesetzt, da die Ergebnisse der Untersuchung nur vom Rang, aber nicht vom Trägheitsindex der jeweiligen quadratischen Grundformen abhängen.

Nun sei $r = 1$. Dann treten an die Stelle der $n-1$ linearen homogenen Orthogonalitätsbedingungen für $\mathfrak{a}_1 \perp \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ für n Komponenten $a_{11}, a_{12}, \ldots, a_{1n}$ die gleiche Anzahl von Bedingungen für nurmehr $n-1$ Komponenten $a_{11}, a_{12}, \ldots, a_{1,n-1}$. Da die triviale Lösung $\mathfrak{a}_1 = 0$ ausgeschlossen werden muß (wir behandeln stets den allgemeinen Fall), ist notwendig eine dieser Bedingungsgleichungen von den übrigen $n-2$ abhängig und wir bekommen insgesamt eine Bedingung weniger und daher für die Parameterzahl der Hauptgruppe des einfach singulären konformeuklidischen $E_n^{(1)}$

$$s = n^2 + 1 - \frac{n(n-1)}{2} - 1 = n^2 + 1 - \frac{n(n-1)}{2} - \frac{1(1+1)}{2} = \frac{n(n+1)}{2} + 2\ ^{6)}.$$

Für $r = 2$ ergeben sich aus den $n-1$ linear homogenen Gleichungen für $\mathfrak{a}_1 \perp \mathfrak{a}_2, \ldots, \mathfrak{a}_n$ höchstens $n-3$ linear unabhängige, da $\mathfrak{a}_1$ jetzt nur noch durch $n-2$ Komponenten $a_{11}, a_{12}, \ldots, a_{1,n-2}$ gegeben ist. Aber auch unter den $n-2$ linear homogenen Orthogonalitätsbedingungen für $\mathfrak{a}_2 \perp \mathfrak{a}_3, \ldots, \mathfrak{a}_n$ gibt es höchstens $n-3$ unabhängige, da $\mathfrak{a}_2 = \{a_{21}, \ldots, a_{2,n-2}\}$ ist und triviale Lösungen $\mathfrak{a}_1 = \mathfrak{a}_2 = 0$ ausgeschlossen werden müssen. Somit ergibt sich für die Parameterzahl der Hauptgruppe des zweifach singulären $E_n^{(2)}$ $(n \geqq 3)$

$$s = n^2 + 1 - \frac{n(n-1)}{2} - 3 = n^2 + 1 - \frac{n(n-1)}{2} - \frac{2(2+1)}{2} = \frac{n(n+1)}{2} + 4.$$

Schließlich erhält man allgemein

$$(7) \quad s = n^2 + 1 - \frac{n(n-1)}{2} - \frac{r(r+1)}{2} = \frac{n(n+1) + r(r+1) + 2}{2},$$
$$r = 0, 1, 2, \ldots, n-1$$

wesentliche Parameter und damit das Ergebnis:

(I) *Die Hauptgruppe eines r-fach singulären konformeuklidischen Raumes* $E_n^{(r)}$ *ist eine* $G_{\frac{n(n+1)+r(r+1)+2}{2}}$; $r = 0, 1, 2, \ldots, n-1$.

Die „absolute M_{n-2}^2" eines totalsingulären $E_n^{(n)}$ ist mit der $(n-1)$-dimensionalen Fernhyperebene x_0 des $E_n^{(n)}$ identisch, deren automorphe Kollineationen durch die Transformationen

$$\varrho x_0 = a_{10} x_0', \quad \varrho x_i = \sum_{k=0}^{n} a_{ik} x_k', \quad \varrho \neq 0, \quad |a_{lk}| \neq 0, \quad \begin{array}{l} i = 1, 2, \ldots, n, \\ l, k = 0, 1, 2, \ldots, n, \end{array}$$

der $n(n+1)$-parametrigen affinen Gruppe $A_{n(n+1)}$ gegeben sind. Daher gilt:

(II) *Die Hauptgruppe eines totalsingulären* $E_n^{(n)}$ *ist die affine Gruppe* $A_{n(n+1)}$.

Für den Defekt $r = n-1$ ergibt sich:

(III) *Die Hauptgruppe eines* $(n-1)$*-fach singulären* $E_n^{(n-1)}$ *ist die biaffine Gruppe*

$$B_{n^2+1}.$$

In konformeuklidischen Räumen gerader Dimensionszahl $n = 2m$ erscheint noch der Defekt $r = m$ bemerkenswert. In diesem Falle gilt:

(IV) *Die Hauptgruppe eines m-fach singulären* $E_{2m}^{(m)}$ *ist die halbaffine Gruppe*

$$H_{\frac{5m^2+3m+2}{2}}.$$

---

[6]) Vgl. z. B. *K. Strubecker*, Die Geometrie des isotropen Raumes und einige ihrer Anwendungen, Jahresbericht D. M. V. 48 (1938), 236–257; Beiträge zur Geometrie des isotropen Raumes, Journal f. reine und angew. Math. 178 (1938), 135–173.

§ 2. Tabellierung für $1 \leqq n \leqq 8$ und $0 \leqq n \leqq 8$.

	$E_n^{(0)}$	$E_n^{(1)}$	$E_n^{(2)}$	$E_n^{(3)}$	$E_n^{(4)}$	$E_n^{(5)}$	$E_n^{(6)}$	$E_n^{(7)}$	$E_n^{(8)}$
$n=1$	G_2	A_2							
$n=2$	G_4	B_5	A_6						
$n=3$	G_7	G_8	B_{10}	A_{12}					
$n=4$	G_{11}	G_{12}	H_{14}	B_{17}	A_{20}				
$n=5$	G_{16}	G_{17}	G_{19}	G_{22}	B_{26}	A_{30}			
$n=6$	G'_{22}	G_{23}	G_{25}	H_{28}	G_{32}	B_{37}	A_{42}		
$n=7$	G_{29}	G_{30}	G'_{32}	G_{35}	G_{39}	G_{44}	B_{50}	A_{56}	
$n=8$	G_{37}	G_{38}	G_{40}	G_{43}	H_{47}	G_{52}	G_{58}	B_{65}	A_{72}

Bei konstanter Dimensionszahl n vermehrt der Defekt $r \leqq n-1$ die Parameterzahl um $\frac{r(r+1)}{2}$, der Defekt $r=n$ jedoch um $\frac{n^2+n-2}{2}$. Bei konstantem Defekt $r \leqq n-1$ steigt die Anzahl der Parameter um $n+1$, wenn man von der Dimension n zur Dimension $n+1$ übergeht, dagegen um $n+2$, wenn $r=n$ ist. Es gilt $B_5 = H_5$.

§ 3. Einbettungsketten und induzierte Hauptgruppen.

Im Gegensatz zur schwierigen Einbettungstheorie (regulärer oder singulärer) Riemannscher Mannigfaltigkeiten in ebensolche andere (geeigneter Dimension) verläuft die Einbettungstheorie regulärer oder singulärer konformeuklidischer $E_n^{(r)}$ in ebensolche $E_N^{(\varrho)}$ algebraisch elementar. Sind $\mathfrak{e}_i (i = 1, 2, \ldots, n)$ n linear unabhängige Basisvektoren, welche einen $E_n^{(r)}$ aufspannen, so ist die Komponentenmatrix $\| e_{ik} \|$ dieser Vektoren vom Rang n, die Gramsche Matrix $\| \mathfrak{e}_i \mathfrak{e}_k \|$ ihrer Skalarprodukte jedoch vom Rang $n-r$. Soll dann $E_n^{(r)}$ in $E_N^{(\varrho)}$ eingebettet werden, so gibt es notwendig $N-n$ weitere Basisvektoren $\mathfrak{f}_1, \mathfrak{f}_2, \ldots, \mathfrak{f}_{N-n}$, welche zusammen mit $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n$ den $E_N^{(\varrho)}$ aufspannen. Für die Determinante der Vektoren $\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{f}_{N-n}$ und ebenso für ihr Quadrat (d. h. für ihre Gramsche Determinante) gilt dann notwendig

$$(\mathfrak{e}_1, \mathfrak{e}_2, \ldots, \mathfrak{e}_n, \mathfrak{f}_1 \ldots, \mathfrak{f}_{N-n})^2 \neq 0 \tag{8}$$

und die Einbettung wird unmöglich, wenn die Bedingung (8) mit den Voraussetzungen über den Rang der Gramschen Matrix $\| \mathfrak{e}_i \mathfrak{e}_k \|$ in Widerspruch steht[7]). So ist z. B. die

[7]) Zur Einbettungstheorie singulärer Riemannscher μ-dimensionaler Mannigfaltigkeiten ($1 \leqq \mu \leqq n+1$) in einen komplexen euklidischen R_{n+2} vgl. *M. Pinl*, Zur integrallosen Darstellung n-dimensionaler isotroper Mannigfaltigkeiten in R_{n+2}, Math. Zeitschrift 42 (1937), 337—354.

Einbettung $E_3^{(2)}$ in $E_4^{(0)}$ unmöglich, denn für jede Wahl des Vektors $\mathfrak{f}$ gilt, wenn $\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3$ Basisvektoren von $E_3^{(2)}$ darstellen,

$$(\mathfrak{e}_1, \mathfrak{e}_2, \mathfrak{e}_3, \mathfrak{f})^2 = \begin{vmatrix} \mathfrak{e}_1^2, & \mathfrak{e}_1\mathfrak{e}_2, & \mathfrak{e}_1\mathfrak{e}_3, & \mathfrak{e}_1\mathfrak{f} \\ \mathfrak{e}_2\mathfrak{e}_1, & \mathfrak{e}_2^2, & \mathfrak{e}_2\mathfrak{e}_3 & \mathfrak{e}_2\mathfrak{f} \\ \mathfrak{e}_3\mathfrak{e}_1, & \mathfrak{e}_3\mathfrak{e}_2, & \mathfrak{e}_3^2, & \mathfrak{e}_3\mathfrak{f} \\ \mathfrak{f}\mathfrak{e}_1, & \mathfrak{f}\mathfrak{e}_2, & \mathfrak{f}\mathfrak{e}_3, & \mathfrak{f}^2 \end{vmatrix} \equiv 0 \ \{\mathfrak{f}\},$$

wie etwa die Entwicklung nach den Elementen der vierten Kolonne zeigt, da auf $E_3^{(2)}$ nach Voraussetzung alle Unterdeterminanten zweiter Ordnung aus $\| \mathfrak{e}_i \mathfrak{e}_k \|$ verschwinden. Für $1 \leqq n \leqq 8$ ergeben sich z. B. die folgenden Ketten eingebetteter Räume, deren Dimensionsverhältnisse auch schon aus Untersuchungen von Herrn J. Lense bekannt geworden sind[8]):

| | | |

zu einem konformeuklidischen $E_m^{*(0)}$ wird. Durch weitere Einbettungen in dieser Seitenkette der $E_m^{*(0)}$, $E_{m-1}^{*(0)}$, $E_{m-1}^{*(1)}$, $E_{m-2}^{*(2)}, \ldots$ gelangt man mit entsprechend verminderter Dimensionszahl wiederum zu einer affinen Hauptgruppe $A_{p(p+1)}$ usw. Der einmal iterierten Seitenkette der Einbettungen

$$E_8^{(0)},\ E_4^{(4)};\ E_4^{*(0)},\ E_2^{*(2)};\ E_2^{**(0)},\ E_1^{**(1)}$$

entspricht so z. B. die Gruppenfolge

G_{37} (aequiform), A_{20} (affin), G_{11} (aequiform), A_6 (affin), G_4 (aequiform), A_2 (affin).

Eingegangen 12. Februar 1959.

Distribution of Bordered Persymmetric Matrices in a Finite Field.

By *David E. Daykin*, Reading (England).

1. Introduction.

A matrix $M = [\alpha_{i,j}]$ is said to be persymmetric if $\alpha_{i,j} = \alpha_{r,s}$ whenever $i + j = r + s$. The term *persymmetric* was coined by Sylvester, and persymmetric determinants in particular have been studied by several authors[1]). In this paper I solve some problems concerning the numbers of persymmetric matrices with elements in a finite field which have certain properties. The problems are anologous to those which have been dealt with in [2] and [3] by L. Carlitz and John H. Hodges for other kinds of matrices, such as hermitian and symmetric.

My thanks are due to Professor R. Rado for the advice and encouragement which he has given me during the preparation of this paper.

2. Statement of results.

Let $GF(q)$ denote a finite field of q elements. Except where indicated, small greek letters $\alpha, \beta, \gamma, \ldots$ will denote elements of $GF(q)$, italic capital letters $A, B, C, \ldots$ will denote matrices with elements in $GF(q)$, and small roman letters $a, b, c, \ldots$ will denote non-negative integers.

Theorem 1. *The number $\mathfrak{P}(m, n, k)$ of persymmetric $m \times n$ matrices over $GF(q)$ of rank k is given by*

$$\mathfrak{P}(m, n, k) = \begin{cases} 1 & \text{if } k = 0, \quad (2.1) \\ q^{2(k-1)}(q^2 - 1) & \text{if } 1 \leqq k < \min(m, n), \quad (2.2) \\ q^{2(k-1)}(q^{|m-n|+1} - 1) & \text{if } 1 \leqq k = \min(m, n), \quad (2.3) \\ 0 & \text{in all other cases.} \quad (2.4) \end{cases}$$

An $m \times n$ persymmetric matrix M may be represented by $M = [\alpha_{i+j-1}]$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, and for each such matrix we define a number $\Phi(M)$ as follows. If there is a least integer u such that $\alpha_u \neq 0$ then $\Phi(M) = u$, otherwise $\Phi(M) = m + n$. We then have

Theorem 2. *The number $\mathfrak{Q}(m, n, k : u)$ of persymmetric $m \times n$ matrices M over $GF(q)$ of rank k such that $\Phi(M) = u$ is given by*

$$\mathfrak{Q}(m, n, k : u) = \begin{cases} (q-1)\, q^u\, \mathfrak{P}(m-u, n-u, k-u) & \text{if } 1 \leqq u < \min(m, n), \\ (q-1)\, q^{m+n-u-1} & \text{if } 1 \leqq \min(m, n) \leqq u < \max(m, n) \text{ and } k = \min(m, n), \\ (q-1)\, q^{m+n-u-1} & \text{if } \max(m, n) \leqq u < m + n \text{ and } k = m + n - u, \\ 1 & \text{if } u = m + n \text{ and } k = 0, \\ 0 & \text{in all other cases.} \end{cases}$$

[1]) A bibliography is given in [1].

Let A be a fixed, square, persymmetric matrix over $GF(q)$ of order m and rank k. In theorem 3 we count the number $\mathfrak{R}(A:r,s,h)$ of ways in which A may be bordered on the right by s columns and below by r rows of elements of $GF(q)$ so that the resulting matrix is persymmetric and has rank $k+h$. We will find that $\mathfrak{R}(A:r,s,h)$ depends upon the number $\Psi(A)$ defined in the following way. If there is a largest integer v such that the minor contained in the first v rows and v columns of A is non-singular then $\Psi(A)=v$, otherwise $\Psi(A)=0$.

Theorem 3. *If A is a square, persymmetric matrix over $GF(q)$ of order m and rank k, and if $v=\Psi(A)$ then for all non-negative integers r, s, h we have*

$$\mathfrak{R}(A:r,s,h)=\begin{cases} \mathfrak{Q}(m-v+r,\, m-v+s,\, k-v+h:2m-k-v)/(q-1)\, q^{k-v-1} & \text{if } v<k, \quad (2.5)\\ q^{a} \sum\limits_{2(m-v)\leq i\leq 2(m-v)+r+s} \mathfrak{Q}(m-v+r,\, m-v+s,\, h:i) & \text{if } v=k, \quad (2.6)\end{cases}$$

where a is $0, 1$ or $r+s$ according to the rule: $a=0$ if $0\leq k<m$; $a=1$ if $k=m$ and $rs>0$; $a=r+s$ if $k=m$ and $rs=0$.

For each element β of $GF(q)$ and each positive integer m let $\mathfrak{S}(m,\beta)$ denote the number of square, persymmetric matrices M of order m over $GF(q)$ such that $\det(M)=\beta$. Then it follows from theorem 1 that

$$\mathfrak{S}(m,0)=q^{2m-1}-\mathfrak{P}(m,m,m)=q^{2m-1}-q^{2(m-1)}(q-1)=q^{2(m-1)}.$$

Our last theorem determines $\mathfrak{S}(m,\beta)$ for the case in which $\beta\neq 0$. We need the function Θ defined as follows. If γ is a non-zero element of $GF(q)$ and $k_1, k_2, \ldots, k_t$ are any integers then

$$\Theta(k_1,\ldots,k_t:\gamma)=\begin{cases} d & \text{if } d \,\Big|\, \dfrac{q-1}{r},\\ 0 & \text{otherwise,}\end{cases}$$

where d is the highest common factor $(k_1,\ldots,k_t,q-1)$ and r is the multiplicative order of γ in $GF(q)$.

Theorem 4. *If m is a positive integer and β is a non-zero element of $GF(q)$ then*

$$\mathfrak{S}(m,\beta)=q^{m-1}\Big\{\Theta(m:\beta(-\;)^{\binom{m}{2}})+(q-1)\sum_{1\leq u<m}\Theta(m-u,2m-u:\beta(-1)^{\binom{u}{2}+\binom{m-u}{2}})$$
$$+(q-1)^2\sum_{\substack{1\leq u<m\\ 1\leq v<m-u}}\Theta(m-u-v,\,2m-2u-v,\,2m-u:\beta(-1)^{\binom{u}{2}+\binom{v}{2}+\binom{m-u-v}{2}})+\cdots\Big\}. \quad (2.7)$$

3. Reduction Process.

Let M be a persymmetric $m\times n$ matrix over $GF(q)$ and put $u=\Phi(M)$. We shall first suppose that $1\leq u<\min(m,n)$. Then if $M=[\alpha_{i+j-1}]$, $1\leq i\leq m$, $1\leq j\leq n$, by definition of $\Phi(M)$ we have $\alpha_u\neq 0$. We now use the elements of M to define inductively a sequence $\theta_1, \theta_2, \ldots, \theta_{m+n-u}$. First we put $\theta_1=\alpha_u^{-1}$ so that

$$\alpha_u\,\theta_1=1. \quad (3.1)$$

Then if t is one of the integers $2, 3, \ldots, m+n-u$ and $\theta_1, \theta_2, \ldots, \theta_{t-1}$ have been defined we choose θ_t to satisfy the equation

$$\alpha_u\theta_t+\alpha_{u+1}\theta_{t-1}+\cdots+\alpha_{u+t-1}\theta_1=0,\quad 2\leq t\leq m+n-u. \quad (3.2)$$

Next put

$$(3.3)\qquad F=\begin{bmatrix}\theta_1 & & & \\ \theta_2 & \theta_1 & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \theta_m & \theta_{m-1} & \cdots & \theta_2 \quad \theta_1\end{bmatrix},\quad G=\begin{bmatrix}\theta_1 & \theta_2 & \cdots & \theta_n \\ & \theta_1 & \cdot & \theta_{n-1} \\ & & \cdot & \cdot \\ & & \cdot & \theta_2 \\ & & & \theta_1\end{bmatrix},$$

$$H=\begin{bmatrix}\theta_1 & & & \\ \theta_2 & \theta_1 & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \\ \theta_1 & \theta_2 & \cdots & \theta_{u-1}\ \theta_u\end{bmatrix},\quad \text{and } K=\begin{bmatrix}\theta_{u+2} & \theta_{u+3} & \cdots & \theta_{n+1} \\ & & & \cdot \\ & & & \cdot \\ & & & \cdot \\ & & & \theta_{m+n-u}\end{bmatrix},$$

where F, G, H and K are of types $m\times m$, $n\times n$, $u\times u$ and $(m-u)\times(n-u)$ respectively, and H and K are persymmetric. It will be seen that F, G and H are non-singular.

Lemma 1.

$$FMG=\begin{bmatrix}H & \cdot \\ \cdot & -K\end{bmatrix}.$$

Proof. Let $FMG=[\beta_{i,j}]$, $1\leqq i\leqq m$, $1\leqq j\leqq n$. Then

$$(3.4)\qquad \beta_{i,j}=\sum_{1\leqq r\leqq i}\theta_{i-r+1}\Big\{\sum_{1\leqq s\leqq j}\alpha_{r+s-1}\,\theta_{j-s+1}\Big\}.$$

Suppose first that $i\leqq u$. Then since $\alpha_1=\alpha_2=\cdots=\alpha_{u-1}=0$, and by virtue of equations (3.1) and (3.2), we have

$$\sum_{1\leqq s\leqq j}\alpha_{r+s-1}\,\theta_{j-s+1}=\begin{cases}1 & \text{if } r+j-1=u,\\ 0 & \text{otherwise.}\end{cases}$$

Applying this result to equation (3.4) we obtain for $i\leqq u$

$$(3.5)\qquad \beta_{i,j}=\begin{cases}\theta_{i+j-u} & \text{if } 1\leqq u+1-j\leqq i,\\ 0 & \text{otherwise.}\end{cases}$$

Now suppose that $i>u$. In this case we rewrite equation (3.4) in the form

$$(3.6)\qquad \beta_{i,j}=\sum_{1\leqq s\leqq j}\theta_{j-s+1}\Big\{\sum_{1\leqq r\leqq i}\alpha_{r+s-1}\,\theta_{i-r+1}\Big\},$$

and by reasoning as above we obtain

$$(3.7)\qquad \beta_{i,j}=0 \text{ for } i>u \text{ and } j\leqq u.$$

Finally suppose that $i>u$ and $j>u$. Then equation (3.6) may be written as

$$\beta_{i,j}=\beta_{i,u}+\sum_{u+1\leqq s\leqq j}\theta_{j-s+1}\Big\{\Big(\sum_{u-s+1\leqq r\leqq i}\alpha_{r+s-1}\,\theta_{i-r+1}\Big)-\Big(\sum_{u-s+1\leqq r\leqq 0}\alpha_{r+s-1}\,\theta_{i-r+1}\Big)\Big\},$$

and so, using (3.7) and (3.2) this equation becomes simply

$$\begin{aligned}\beta_{i,j}&=-\sum_{u+1\leqq s\leqq j}\theta_{j-s+1}\Big(\sum_{u-s+1\leqq r\leqq 0}\alpha_{r+s-1}\,\theta_{i-r+1}\Big)\\ &=-\sum_{u-j+1\leqq r\leqq 0}\theta_{i-r+1}\Big\{\sum_{u-r+1\leqq s\leqq j}\alpha_{r+s-1}\,\theta_{j-s+1}\Big\}.\end{aligned}$$

By virtue of (3.1) and (3.2) the equation now reduces to

$$(3.8)\qquad \beta_{i,j}=-\theta_{i+j-u} \text{ for } i,j>u.$$

The lemma follows from relations (3.5), (3.7) and (3.8).

For each persymmetric matrix M we now define a set $\{U, \theta, V\}$ of three persymmetric matrices U, θ and V. If M is of type $m \times n$ and $\Phi(M) < \min(m, n)$ then $U = H$, $\theta = \theta_{u+1}$ and $V = -K$ where H, θ_{u+1} and K are as defined above. If on the other hand $\Phi(M) \geqq \min(m, n)$ then $U = M$ whilst $\theta = V = E$ where E denotes the empty 0×0 persymmetric matrix. It will be convenient to write $\mathfrak{A}(M) = \{U, \theta, V\}$.

Lemma 2. *If M is a persymmetric matrix and $\mathfrak{A}(M) = \{U, \theta, V\}$ then*

$$\operatorname{rank}(V) = \operatorname{rank}(M) - \operatorname{rank}(U),$$

and

$$\Psi(V) = \Psi(M) - \Psi(U). \tag{3. 9}$$

Moreover if M is square of order m, say $M = [\alpha_{i+j-1}]$, $1 \leqq i, j \leqq m$, and if $\Phi(M) = u$, $1 \leqq u < m$, then

$$\det(V) = (-1)^{\binom{u}{2}} \alpha_u^{u-2m} \cdot \det(M).$$

Proof. We will only deal with equation (3. 9) because the other two equations follow immediately from the definitions and lemma 1. Put $u = \Phi(M)$ and let the type of M be $m \times n$. If $u \geqq \min(m, n)$ then $U = M$ whilst V is an empty matrix, and therefore (3. 9) holds. Suppose now that $u < \min(m, n)$ and let x be any integer, $u < x \leqq \min(m,n)$. Further let X be the square submatrix contained in the first x rows and columns of the matrix M, and put $\mathfrak{A}(X) = \{Y, \sigma, Z\}$. Now the definitions of $\mathfrak{A}(M)$ and $\mathfrak{A}(X)$ depend essentially on the relations (3. 1), (3. 2) and (3. 3), and hence it follows that $Y = U$, $\sigma = \theta$ and Z is the submatrix contained in the first $x - u$ rows and columns of the matrix V. Moreover since the matrix Y is non-singular, the equation $\operatorname{rank}(Z) = \operatorname{rank}(X) - \operatorname{rank}(Y)$ shows that Z is non-singular if and only if X is non-singular. If $x = \Psi(M)$ then X is non-singular, but if $x > \Psi(M)$ then X is singular. Hence $\Psi(V) = \Psi(M) - u$ and since $u = \Psi(U)$, this ends the proof.

4. Proof of theorem 2.

Let m, n, k and u be any non-negative integers. Also let M be any persymmetric matrix over $GF(q)$ of type $m \times n$ and rank k such that $\Phi(M) = u$. We wish to find the number $\mathfrak{Q}(m, n, k : u)$ of matrices M. It is sufficient for us to assume here that $1 \leqq u < \min(m, n)$ because the expressions given in theorem 2 for $\mathfrak{Q}(m, n, k : u)$ are easily obtained by means of a diagram in the other cases.

Let $\mathfrak{A}(M) = \{U, \theta, V\}$. Then U is a lower diagonal, persymmetric, non-singular $u \times u$ matrix, whilst V is a persymmetric matrix of type $(m - u) \times (n - u)$ and rank $k - u$. Moreover every choice of a lower diagonal, persymmetric, non-singular $u \times u$ matrix H, an element θ_{u+1}, and a persymmetric matrix K of type $(m - u) \times (n - u)$ and rank $k - u$ will lead, by means of lemma 1 and the linear relations (3. 1), (3. 2) and (3. 3) to a choice of a matrix M. Hence the number of matrices M is the product of the numbers of choices for H, θ_{u+1} and K, i. e.

$$\mathfrak{Q}(m, n, k : u) = (q - 1)\, q^{u-1} \cdot q \cdot \mathfrak{P}(m - u, n - u, k - u).$$

This ends our proof.

5. Proof of theorem 1.

Equations (2. 1) and (2. 4) are trivial. Also by the definitions we have

$$\mathfrak{P}(m, n, k) = \sum_{1 \leqq u \leqq m+n} \mathfrak{Q}(m, n, k : u). \tag{5. 1}$$

Suppose that $1 \leqq k < \min(m, n)$. Then by virtue of theorem 2 equation (5. 1) becomes

$$\mathfrak{P}(m, n, k) = \sum_{1 \leqq u \leqq k} (q-1)\, q^u\, \mathfrak{P}(m-u, n-u, k-u) + (q-1)\, q^{k-1}$$
$$= (q-1)\, \{q^{k-1} + q^k + \sum_{1 \leqq u < k} q^u\, \mathfrak{P}(m-u, n-u, k-u)\}.$$

In particular if $k = 1$ then

$$\mathfrak{P}(m, n, k) = q^2 - 1,$$

so that (2. 2) holds for $k = 1$. Let t be any integer, $2 \leqq t < \min(m, n)$, and assume that (2. 2) holds for $1 \leqq k < t$. Then

$$\mathfrak{P}(m, n, t) = (q-1)\, \{q^{t-1} + q^t + \sum_{1 \leqq u < t} q^u q^{2(t-u-1)}(q^2-1)\} = q^{2(t-1)}(q^2-1),$$

and equation (2. 2) follows by induction.

Finally suppose that $1 \leqq k = \min(m, n)$. Without loss of generality we may assume that $m \leqq n$ and then $k = m$. Clearly $\mathfrak{P}(1, n, 1) = q^n - 1$ so that equation (2. 3) holds for $k = 1$. Let t be any integer, $t \geqq 2$, and assume that (2.3) holds for $k = 1, 2, \ldots, t-1$. Applying theorem 2 to equation (5. 1) we obtain

$$\mathfrak{P}(t, n, t) = \sum_{1 \leqq u < t} (q-1)\, q^u\, \mathfrak{P}(t-u, n-u, t-u) + \sum_{t \leqq u \leqq n} (q-1)\, q^{t+n-u-1},$$

so that by our induction hypothesis

$$\mathfrak{P}(t, n, t) = \sum_{1 \leqq u < t} (q-1)\, q^u \cdot q^{2(t-u-1)}(q^{n-t+1} - 1) + q^n - q^{t-1}$$
$$= (q^{n-t+1} - 1)\, (q^{2t-2} - q^{t-1} + q^{t-1}) = q^{2(t-1)}(q^{|t-n|+1} - 1),$$

and equation (2. 4) follows by induction. This proves theorem 1.

6. Proof of theorem 3.

Let A be a fixed, square, persymmetric matrix over $GF(q)$ of order m and rank k. Further let r, s, h be any non-negative integers, and let B, C, D be any matrices over $GF(q)$ of types $m \times s$, $r \times m$ and $r \times s$ respectively such that the matrix

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is persymmetric and has rank $k + h$. We wish to find the number $\mathfrak{Q}(A : r, s, h)$ of such matrices M.

Put $A_0 = A$ and let $\mathfrak{A}(A_0) = \{X_1, \theta_1, A_1\}$, $\mathfrak{A}(A_1) = \{X_2, \theta_2, A_2\}$, and so on. If for some non-negative integer i we have $\Psi(A_i) > 0$ then the matrix X_{i+1} is square, of positive order, and non-singular. It therefore follows by lemma 2 that there is a least integer d such that $\Psi(A_d) = 0$. Moreover if $v = \Psi(A_0)$ then

$$\sum_{1 \leqq i \leqq d} \text{order } (X_i) = v,$$

whilst the matrix A_d is square, of order $m - v$ and rank $(A_d) = k - v$.

Next put $M_0 = M$ and let $\mathfrak{A}(M_0) = \{Y_1, \sigma_1, M_1\}$, $\mathfrak{A}(M_1) = \{Y_2, \sigma_2, M_2\}$, and so on. Now if $M_0 = [\alpha_{i+j-1}]$, $1 \leqq i \leqq m + r$, $1 \leqq j \leqq m + s$, then $A_0 = [\alpha_{i+j-1}]$, $1 \leqq i, j \leqq m$. Moreover if $v > 0$ so that $d > 0$, then since the definitions of $\mathfrak{A}(A_0)$ and $\mathfrak{A}(M_0)$ depend essentially on the relations (3. 1), (3. 2) and (3. 3) we have $Y_1 = X_1$ and $\sigma_1 = \theta_1$; further if $M_1 = [\beta_{i+j-1}]$, $1 \leqq i \leqq m + r - \Phi(A_0)$, $1 \leqq j \leqq m + s - \Phi(A_0)$, then $A_1 = [\beta_{i+j-1}]$, $1 \leqq i, j \leqq m - \Phi(A_0)$.

Thus step by step we obtain the relations

$$(6.1)\qquad Y_i = X_i \text{ and } \sigma_i = \theta_i \text{ for } 1 \leqq i < d, \text{ whilst } \sigma_d = \theta_d \text{ if } v < m.$$

Note also that the matrix M_d has rank $k - v + h$, type $(m - v + r) \times (m - v + s)$ and is of the form

$$(6.2)\qquad M_d = \begin{array}{c} \begin{bmatrix} A_d & B_d \\ C_d & D_d \end{bmatrix} \begin{array}{l} \}\, m - v \text{ rows,} \\ \}\, r \text{ rows.} \end{array} \\ \begin{array}{cc} m - v \text{ cols.} & s \text{ cols.} \end{array} \end{array}$$

Now each choie of a matrix M will lead to a sequence $Y_1, \sigma_1, Y_2, \sigma_2, \ldots, Y_d, \sigma_d, M_d$ of persymmetric matrices satisfying (6. 1) and (6. 2), and such that rank $(M_d) = k - v + h$, and σ_d is of type 1×1. Vice versa each choice of a sequence $Y_1, \sigma_1, Y_2, \sigma_2, \ldots, Y_d, \sigma_d, M_d$ of persymmetric matrices satisfying (6. 1) and (6. 2), and such that rank $(M_d) = k - v + h$, whilst σ_d is of type 1×1, will lead to a possible choice of a matrix M, except in the case when M_d has no rows or no columns, for in this case all information concerning the matrices B, C and D has been lost in the reduction. The matrix M_d has no rows or no columns when $(m - v + r)\ (m - v + s) = 0$, that is when $rs = 0$ and $m = v$ so that $k = v$. Clearly in all other cases the number $\mathfrak{Q}(A : r, s, h)$ of choices for the matrix M is the number of choices for σ_d and M_d.

Suppose first that $rs = 0$ and $m = v = k$ so that the matrix M_d has no rows or no columns. Then the matrix A is non-singular and because $rs = 0$ any matrix formed by adding r rows and s columns to A will have rank k. Thus in this case the number $\mathfrak{Q}(A : r, s, h)$ is q^{r+s} if $h = 0$, but 0 if $h > 0$, and it follows by theorems 2 and 1 that equation (2. 6) holds. We will now deal with the cases in which the matrix M_d is not empty.

Suppose that $v = k$ and $0 \leqq k < m$. Then rank $(M_d) = h$ and since rank $(A_d) = k - v$ the matrix A_d has no element which is not zero. Hence it may be seen from equation (6. 2) that the number of choices for the matrix M_d is

$$\sum_{2(m-v) \leqq i \leqq 2(m-v)+r+s} \mathfrak{Q}(m - v + r, m - v + s, h : i).$$

We have only one choice for σ_d because relation (6. 1) shows that $\sigma_d = \theta_d$, and so equation (2. 6) holds.

If $v = k = m$ and $rs > 0$ then the matrices A_d and θ_d are empty. Hence we have q choices for σ_d whilst for M_d we have $\mathfrak{P}(r, s, h)$ choices. Thus equation (2. 6) also holds for it's last case, by virtue of equation (5. 1).

Next suppose that $v < k$. Then $v < m$ and by equation (6. 1) we have $\sigma_d = \theta_d$. Also A_d has at least one non-zero element and so $\Phi(A_d) = \Phi(M_d)$. Let $u = \Phi(A_d)$, then since rank $(A_d) = k - v$ it is easy to see from a detailed diagram of (6. 2) that $u = 2m - k - v$. If in particular $u \geqq \min(m - v + r, m - v + s)$, then the diagram will also show that the number of choices for M_d is q^{r+s} if $h = r + s$ and 0 if $h \neq r + s$. Since by theorem 2 we have for this case

$$\mathfrak{Q}(m - v + r, m - v + s, k - v + h : 2m - k - v) = \begin{cases} (q-1)\, q^{r+s+k-v-1} & \text{if } h = r + s, \\ 0 & \text{otherwise,} \end{cases}$$

it follows that (2. 5) holds. Now suppose on the other hand that $1 \leqq u < \min(m - v + r, m - v + s)$. Let $\mathfrak{A}(M_d) = \{H, \varrho, K\}$, and let $H = [\gamma_{i+j-1}]$, $1 \leqq i, j \leqq u$. Then we have $\gamma_1 = \gamma_2 = \cdots = \gamma_{u-1} = 0$ whilst $\gamma_u, \gamma_{u+1}, \ldots, \gamma_{u+k-v-1}$ are determined by A_d. Also the matrix K is of type $(m - v + r - u) \times (m - v + s - u)$ and rank $k - v + h - u$.

The number of choices for M_d is the number of choices for H, ϱ and K. We are only free to choose the elements $\gamma_{u+k-v}, \ldots, \gamma_{2u-1}$ of H, so that for H we have q^{u-k+v} choices. For ϱ we have q choices and for K we have

$$\mathfrak{P}(m-v+r-u, m-v+s-u, k-v+h-u)$$

choices. By theorem 2 we are given that

$$\mathfrak{Q}(m-v+r, m-v+s, k-v+h:u)=(q-1)q^u\mathfrak{P}(m-v+r-u, m-v+s-u, k-v+h-u),$$

and so once again equation (2. 5) holds.

This proves theorem 3.

7. Proof of theorem 4.

Let $\mathfrak{J}$ denote the set of all non-zero elements of $GF(q)$.

Lemma 3. *If $\gamma \in \mathfrak{J}$ and $k_1, \ldots, k_s$ are any integers with highest common factor k then the number of solutions in $GF(q)$ of the equation $\xi^k = \gamma$ is given by $\Theta(k_1, \ldots, k_s : \gamma)$.*

Proof. Let η be a generator of the cyclic multiplicative group $\mathfrak{J}$. Then for each $\xi \in \mathfrak{J}$ there is exactly one integer x, $0 \leqq x < q-1$, such that $\eta^x = \xi$. Suppose that $\xi^l = \gamma$. Then it follows that the number of solutions in $GF(q)$ of the equation $\xi^k = \gamma$ is precisely the number of solutions of the congruence $kx \equiv l \pmod{q-1}$. This congruence is soluble if and only if $d \mid l$, where $d = (k, q-1) = (k_1, \ldots, k_s, q-1)$, and if it is soluble then it has just d solutions[2]). Now $d \mid l$ if and only if $d \mid (l, q-1)$ and since $\dfrac{q-1}{(l, q-1)}$ is the multiplicative order r of γ, the lemma now follows.

Lemma 4. *If $\gamma \in \mathfrak{J}$ and $s, k_1, k_2, \ldots, k_s$ are any integers then*

$$\sum_{\alpha \in \mathfrak{J}} \Theta(k_1, \ldots, k_s : \gamma\alpha^s) = (q-1)\, \Theta(k_1, \ldots, k_s, s : \gamma).$$

Proof. Firstly if $k = (k_1, \ldots, k_s)$, then by definition $\Theta(k_1, \ldots, k_s : \gamma\alpha^s) = \Theta(k : \gamma\alpha^s)$. Also we have

$$\sum_{\alpha \in \mathfrak{J}} \Theta(k : \gamma\alpha^s) = \sum_{\xi \in \mathfrak{J}} \Theta(-s : \gamma\xi^{-k}),$$

because each side of this equation is the number of pairs $\alpha, \xi \in \mathfrak{J}$ such that $\xi^k = \gamma\alpha^s$. Next suppose that $s = kp + r$. Then each solution λ of the equation $\xi^k = \gamma\alpha^s$ yields a solution $\lambda\alpha^{-p}$ of the equation $\xi^k = \gamma\alpha^r$, and vice versa. Hence

$$\Theta(k : \gamma\alpha^s) = \Theta(k : \gamma\alpha^r) \quad \text{if } s = kp + r.$$

Without loss of generality we may now assume that $|s| > |k|$. Let $r_0 = s$ and $r_1 = k$, then Euclid's algorithm provides a sequence of equations $r_0 = r_1 p_1 + r_2, \ldots, r_{t-1} = r_t p_t + 0$, where $r_t = (r_0, r_1) = (k_1, \ldots, k_s, s)$. Combining all these results we finally obtain

$$\begin{aligned}\sum_{\alpha \in \mathfrak{J}} \Theta(k_1, \ldots, k_s : \gamma\alpha^s) &= \sum_{\alpha \in \mathfrak{J}} \Theta(r_1 : \gamma\alpha^{r_0}) = \sum_{\alpha \in \mathfrak{J}} \Theta(r_1 : \gamma\alpha^{r_0}) \\ &= \sum_{\alpha \in \mathfrak{J}} \Theta(-r_2 : \gamma\alpha^{-r_1}) = \cdots = \sum_{\alpha \in \mathfrak{J}} \Theta(r_t : \gamma\alpha^0) \\ &= (q-1)\, \Theta(k_1, \ldots, k_s, s : \gamma),\end{aligned}$$

and this proves the lemma.

Now let β be a fixed element of $\mathfrak{J}$ and let m be a fixed positive integer. Also let u be any positive integer and suppose that M is any square, persymmetric matrix over $GF(q)$

[2]) For a proof of this statement see [3], theorem 57, p. 51.

of order m such that $\det(M) = \beta$ and $\Phi(M) = u$. Since $\beta \neq 0$ we may assume that $1 \leqq u \leqq m$, and if $M = [\alpha_{i+j-1}]$, $1 \leqq i, j \leqq m$, we will write α for α_u.

Suppose first that $u = m$. Then we have $\beta = \det(M) = \alpha^m(-1)^{\binom{m}{2}}$. Hence the number of matrices M is q^{m-1} times the number of choices for α. Therefore by lemma 3 the number of matrices M is

$$q^{m-1}\Theta\Big(m : \beta(-1)^{\binom{m}{2}}\Big).$$

Next suppose that $1 \leqq u < m$, and let $\mathfrak{A}(M) = \{U, \theta, V\}$. Then by lemma 2 we have $\det(V) = \alpha^{u-2m}\beta(-1)^{\binom{u}{2}}$. The number of matrices M is the number of choices for U, θ and V and hence this number is

$$\sum_{\alpha \in \mathfrak{F}} q^{u-1} \cdot q \cdot \mathfrak{S}\Big(m-u, \alpha^{u-2m}\beta(-1)^{\binom{u}{2}}\Big).$$

From the above remarks we obtain the recurrance relation

$$(7.1)\quad \mathfrak{S}(m, \beta) = q^{m-1}\Theta\Big(m : \beta(-1)^{\binom{m}{2}}\Big) + \sum_{1 \leqq u < m}\Big[\sum_{\alpha \in \mathfrak{F}} q^u \mathfrak{S}\Big(m-u, \alpha^{u-2m}\beta(-1)^{\binom{u}{2}}\Big)\Big].$$

We will now use this recurrance relation to prove theorem 4 by induction upon m. Clearly if $m = 1$ then $\mathfrak{S}(m, \beta) = 1$ and so equation (2. 7) holds in this case. Let t be any integer, $t \geqq 2$, and suppose that equation (2. 7) holds for $m < t$. Then from equation (7. 1) we obtain

$$\mathfrak{S}(t, \beta) = q^{t-1}\Theta\Big(t : \beta(-1)^{\binom{t}{2}}\Big) + \sum_{1 \leqq u < t}\Big[\sum_{\alpha \in \mathfrak{F}} q^u q^{t-u-1}\Big\{\Theta\Big(t-u : \alpha^{u-2t}\beta(-1)^{\binom{u}{2}+\binom{t-u}{2}}\Big)$$
$$+ (q-1)\sum_{1 \leqq v < t-u}\Theta\Big(t-u-v, 2t-2u-v : \alpha^{u-2t}\beta(-1)^{\binom{u}{2}+\binom{v}{2}+\binom{t-u-v}{2}}\Big) + \cdots\Big\}\Big],$$

and by lemma 4 this equation becomes

$$\mathfrak{S}(t, \beta) = q^{t-1}\Big\{\Theta\Big(t : \beta(-1)^{\binom{t}{2}}\Big) + (q-1)\sum_{1 \leqq u < t}\Theta\Big(t-u, 2t-u : \beta(-1)^{\binom{u}{2}+\binom{t-u}{2}}\Big)$$
$$+ (q-1)^2\sum_{\substack{1 \leqq u < t \\ 1 \leqq v < t-u}}\Theta\Big(t-u-v, 2t-2u-v, 2t-u : \beta(-1)^{\binom{u}{2}+\binom{v}{2}+\binom{t-u-v}{2}} + \cdots\Big\}.$$

Theorem 4 now follows by induction.

References.

[1] *J. Muir*, The theory of determinants, I—IV, London (1923).

[2] *L. Carlitz* and *John H. Hodges*, Distribution of bordered symmetric, skew and hermitian matrices in a finite field, Journal für die reine und angewandte Mathematik, **195** (1955/56), 192—201.

[3] *John H. Hodges*, Distribution of bordered matrices in a finite field, Journal für die reine und angewandte Mathematik, **198** (1957), 10—13.

[4] *G. Hardy* and *E. M. Wright*, An introduction to the theory of numbers, Oxford (1954).

Eingegangen 23. Februar 1959.

The Generation of Arithmetical Identities.

By *P. J. McCarthy* in Tallahassee (Florida*).

1. Introduction.

Let K be the field of complex numbers. A single-valued function $f(n, r)$ of the non-negative integral variable n, with values in K, will be called a k-even function of $n \pmod r$ if $f(n, r)$ depends only on $(n, r^k)_k$. Here $(a, b)_k$ is the largest integral k-th power which divides both a and b. Thus, $f(n, r)$ is a k-even function of $n \pmod r$ if for each non-negative integer n,

$$f(n, r) = f((n, r^k)_k, r).$$

The class of all such functions, for a fixed r, will be denoted by E_k.

The class E_1 has been studied in great detail by Cohen in [5] and [7]. By using the characterizations of the class E_1 given in [5] Cohen, in [7], developed systematic procedures which produce many of the classical, and some new, identities involving well-known arithmetical functions. However, some of the classical arithmetical identities apparently fail to result from an application of these procedures: for example, the identity of Gegenbauer, [8], p. 149,

$$\sum_{d \mid r} d^t J_t\left(\frac{r}{d}\right) J_k(d) = J_{t+k}(r). \tag{1}$$

Here t and k are integers and $J_m(r)$ is the Jordan function ([8], p. 147). In this paper we shall show that a generalization of Cohen's work will produce, among others, the identity (1). We remark that the classical derivation of the identities such as (1) made use of the Riemann zeta function: our work and that of Cohen make no use of this function.

In section 2 we give a characterization of the class E_k which is similar to the characterization of the class E_1 given by Cohen. In section 3 we use this characterization to obtain certain arithmetical identities,. In section 4 we give a generalization of a theorem of Anderson and Apostol [1] and use it to obtain further identities. In section 5 we obtain still more identities. We do not give generalizations of each of the identities contained in Cohen's paper [7], but we illustrate by certain examples how these generalizations can be obtained.

2. Characterization of the class E_k.

Let $e(a, b) = \exp \dfrac{2\pi i a}{b}$ and consider the function

$$c_k(n, r) = \sum_{\substack{(x, r^k)_k = 1 \\ x \pmod{r^k}}} e(nx, r^k).$$

*) The Florida State University.

This is the extension of Ramanujan's function $c(n, r) = c_1(n, r)$ which was defined and studied by Cohen in [2], [3], and [4]. As Cohen remarked, [3], Lemma 2, $c_k(n, r)$ belongs to the class E_k. The following sequence of theorems extends to the class E_k Cohen's characterizations of the class E_1 [5].

Theorem 1. *If $f(n, r)$ belongs to the class E_k, then $f(n, r)$ can be written uniquely in the form*

$$f(n, r) = \sum_{d \mid r} \alpha(d)\, c_k(n, d) \tag{2}$$

where

$$\alpha(d) = \frac{1}{r^k} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) c_k\left(\left(\frac{r}{d}\right)^k, \delta\right). \tag{3}$$

Furthermore,

$$\alpha(d) = \frac{1}{r^k \varphi_k(d)} \sum_{m=1}^{rk} f(m, r)\, c_k(m, d), \tag{4}$$

where $\varphi_k(d) = c_k(0, d)$.

Proof. We first show the uniqueness of the representation (2). This will follow if we show that

$$\sum_{d \mid r} \alpha(d)\, c_k(n, d) = 0, \text{ for all } n, \tag{5}$$

implies that $\alpha(d) = 0$ for all divisors d of r. From [3], Theorem 2 we have that if $d_1 \mid r$ and $d_2 \mid r$, then

$$\sum_{d \mid r} c_k\left(\left(\frac{r}{d}\right)^k, d_1\right) c_k\left(\left(\frac{r}{d_2}\right)^k, d\right) = \begin{cases} r^k \text{ if } d_1 = d_2 \\ 0 \text{ otherwise.} \end{cases} \tag{6}$$

Choose a divisor d_2 of r. Multiplying both sides of (6) by $\alpha(d_1)$ and summing over all divisors d_1 of r yields

$$\sum_{d_1 \mid r} \alpha(d_1) \sum_{d \mid r} c_k\left(\left(\frac{r}{d}\right)^k, d_1\right) c_k\left(\left(\frac{r}{d_2}\right)^k, d\right) = r^k \alpha(d_2).$$

Interchanging the order of summation and using (5) gives $r^k \alpha(d_2) = 0$ which implies that $\alpha(d_2) = 0$.

Using (3) we have

$$\sum_{d \mid r} \alpha(d)\, c_k(n, d) = \frac{1}{r^k} \sum_{d \mid r} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) c_k(n, d).$$

If we replace n by $(n, r^k)_k = \left(\frac{r}{d_1}\right)^k$, this is seen to be equal to

$$\frac{1}{r^k} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) \sum_{d \mid r} c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) c_k\left(\left(\frac{r}{d_1}\right)^k, d\right).$$

From (6) we see that the inner sum is 0 unless $d_1 = \delta$, in which case it is equal to r^k. Thus

$$\sum_{d \mid r} \alpha(d)\, c_k(n, d) = f\left(\left(\frac{r}{d_1}\right)^k, r\right) = f((n, r^k)_k, r) = f(n, r).$$

Finally,

$$\frac{1}{r^k \varphi_k(d)} \sum_{m=1}^{r^k} f(m, r)\, c_k(m, d)$$

$$= \frac{1}{r^k \varphi_k(d)} \sum_{\delta \mid r} \sum_{\substack{(x, r^k)_k = 1 \\ x \pmod{\delta}}} f\left(\left(\frac{r}{\delta}\right)^k x, r\right) c_k\left(\left(\frac{r}{\delta}\right)^k x, d\right)$$

$$= \frac{1}{r^k \varphi_k(d)} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) c_k\left(\left(\frac{r}{\delta}\right)^k, d\right) \varphi_k(\delta),$$

where we have used Lemma 1 of [2]. Now, by [4], Lemma 1,

$$c_k\left(\left(\frac{r}{\delta}\right)^k, d\right) \varphi_k(\delta) = c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) \varphi_k(d),$$

and so the above expression is equal to

$$\frac{1}{r^k} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) = \alpha(d).$$

This proves the equivalence of (3) and (4).

Theorem 2. *$f(n, r)$ belongs to the class E_k if and only if it can be written in the form*

$$f(n, r) = \sum_{d^k \mid (n, r^k)_k} g\left(d, \frac{r}{d}\right) \tag{7}$$

where $g(a, b)$ is a single-valued function of the non-negative integral variables a and b with values in K.

Proof. It is obvious that any function having a representation (7) belongs to the class E_k. On the other hand, if $f(n, r)$ is in the class E_k, we have

$$f(n, r) = \sum_{d \mid r} \alpha(d)\, c_k(n, d)$$

$$= \sum_{d \mid r} \alpha(d) \sum_{\delta^k \mid (n, r^k)_k} \mu\left(\frac{d}{\delta}\right) \delta^k$$

by [2], Theorem 2, and so

$$f(n, r) = \sum_{\delta^k \mid (n, r^k)_k} \delta^k \sum_{\substack{d \mid r \\ \delta \mid d}} \alpha(d)\, \mu\left(\frac{d}{\delta}\right)$$

$$= \sum_{\delta^k \mid (n, r^k)_k} \delta^k \sum_{e\delta \mid r} \alpha(\delta e)\, \mu(e).$$

Hence, $f(n, r)$ has a representation (7) with

$$g(a, b) = a^k \sum_{e \mid b} \alpha(ae)\, \mu(e).$$

Theorem 3. *If $f(n, r)$ has a representation (7), then*

$$\alpha(d) = \frac{1}{r^k} \sum_{ed \mid r} g\left(\frac{r}{e}, e\right) e^k.$$

Proof. We have

$$\alpha(d) = \frac{1}{r^k} \sum_{\delta \mid r} f\left(\left(\frac{r}{\delta}\right)^k, r\right) c_k\left(\left(\frac{r}{d}\right)^k, \delta\right)$$

$$= \frac{1}{r^k} \sum_{\delta \mid r} c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) \sum_{D^k \mid ((r/\delta)^k, r^k)_k} g\left(D, \frac{r}{D}\right)$$

$$= \frac{1}{r^k} \sum_{\delta \mid r} c_k\left(\left(\frac{r}{d}\right)^k, \delta\right) \sum_{D\delta \mid r} g\left(D, \frac{r}{D}\right)$$

$$= \frac{1}{r^k} \sum_{\substack{D \mid r \\ De = r}} g\left(D, \frac{r}{D}\right) \sum_{D\delta \mid r} c_k\left(\left(\frac{r}{d}\right)^k, \delta\right)$$

$$= \frac{1}{r^k} \sum_{e \mid r} g\left(\frac{r}{e}, e\right) \sum_{\delta \mid e} c_k\left(\left(\frac{r}{d}\right)^k, \delta\right).$$

By [3], Corollary 2. 2, the inner sum is e^k if $e \mid \frac{r}{d}$ and 0 otherwise. Hence, $\alpha(d)$ is equal to the expression given in the statement of the theorem.

3. Arithmetical identities.

Let $H(a, b)$ be a function of two non-negative integral variables with values in K. Let s be a complex number. Define

$$h_s(n, r) = \sum_{d^k \mid (n, r^k)_k} d^{-ks} H\left(d^k, \frac{r}{d}\right),$$

$$h'_s(n, r) = \sum_{d^k \mid (n, r^k)_k} \left(\frac{r}{d}\right)^{-ks} H\left(\left(\frac{r}{d}\right)^k, d\right).$$

Both of these functions belong to the class E_k.

Theorem 4.

$$h_s(n, r) = \sum_{d \mid r} h'_{s+1}\left(\left(\frac{r}{d}\right)^k, r\right) c_k(n, d).$$

Proof. Let

$$h_s(n, r) = \sum_{d \mid r} \alpha(d)\, c_k(n, r).$$

By Theorem 3

$$\alpha(d) = \frac{1}{r^k} \sum_{ed \mid r} g\left(\frac{r}{e}, e\right) e^k$$

$$= \frac{1}{r^k} \sum_{ed \mid r} \left(\frac{r}{e}\right)^{-ks} H\left(\left(\frac{r}{e}\right)^k, e\right) e^k$$

$$= \sum_{e \mid ((r/d)^k, r^k)_k} \left(\frac{r}{e}\right)^{-k(s+1)} H\left(\left(\frac{r}{e}\right)^k, e\right)$$

$$= h'_{s+1}\left(\left(\frac{r}{d}\right)^k, r\right).$$

Now let $g(r)$ be a function of a single integral variable with values in K, and set

$$G_s(r) = \sum_{d \mid r} d^s g\left(\frac{r}{d}\right),$$

$$\Gamma_k^{(s)}(n, r) = \sum_{d^k \mid (n, r^k)_k} d^s g\left(\frac{r}{d}\right).$$

If we choose $H(a, b) = g(b)$ for all a and b, then

$$h_s(n, r) = \Gamma_k^{(-ks)}(n, r)$$

and

$$h'_{s+1}\left(\left(\frac{r}{d}\right)^k, r\right) = \sum_{d\delta \mid r} \left(\frac{r}{\delta}\right)^{-k(s+1)} g(\delta).$$

But,

$$d^{-k(s+1)} G_{-k(s+1)}\left(\frac{r}{d}\right) = d^{-k(s+1)} \sum_{d\delta \mid r} \delta^{-k(s+1)} g\left(\frac{r}{d\delta}\right)$$
$$= \sum_{d\delta \mid r} \left(\frac{r}{\delta}\right)^{-k(s+1)} g(\delta).$$

Hence, an application of Theorem 4 gives

$$\Gamma_k^{(-ks)}(n, r) = \sum_{d \mid r} d^{-k(s+1)} G_{-k(s+1)}\left(\frac{r}{d}\right) c_k(n, d), \tag{8}$$

or, replacing s by $-s$,

$$\Gamma_k^{(ks)}(n, r) = \sum_{d \mid r} d^{k(s-1)} G_{k(s-1)}\left(\frac{r}{d}\right) c_k(n, d). \tag{9}$$

Example 1. Let $g(r) = 1$ for all r. Then

$$G_s(r) = \sum_{d \mid r} d^s = \sigma_s(r),$$

and if we note that $\sigma_{-s}(r) = \dfrac{\sigma_s(r)}{r^s}\, r$, (8) becomes

$$\Gamma_k^{(-ks)}(n, r) = \frac{1}{r^{k(s+1)}} \sum_{d \mid r} \sigma_{k(s+1)}\left(\frac{r}{d}\right) c_k(n, d).$$

If we now set $n = 0$ we have $\Gamma_k^{(-ks)}(0, r) = \sigma_{-ks}(r)$, and so

$$\sum_{d \mid r} \sigma_{k(s+1)}\left(\frac{r}{d}\right) \varphi_k(d) = r^k \sigma_{ks}(r)$$

(see [7], Corollary 1. 1).

Example 2. For any complex number $s \neq 0$ we set

$$\varphi_s(r) = \sum_{d \mid r} d^s \mu\left(\frac{r}{d}\right). \tag{10}$$

If we let $g(r) = \mu(r)$ for all r, then $G_s(r) = \varphi_s(r)$, and (9) becomes

$$\Gamma_k^{(ks)}(n, r) = \sum_{d \mid r} d^{k(s-1)} \varphi_{k(s-1)}\left(\frac{r}{d}\right) c_k(n, d).$$

If we set $n = 0$, note that $\Gamma_k^{(ks)}(0, r) = \varphi_{ks}(r)$, and set $k(s-1) = t$, this gives

$$\sum_{d \mid r} d^t \varphi_t\left(\frac{r}{d}\right) \varphi_k(d) = \varphi_{t+k}(r). \tag{11}$$

Now, if t is a positive integer then $\varphi_t(r) = J_t(r)$ [6, Theorem 5], and so (11) is a generalization of Gegenbauer's identity (1).

Example 3. For any complex numbers s and t we set

$$g(r) = r^{k(s-1)}\varphi_t(r).$$

Then

$$G_{k(s-1)}(r) = r^{k(s-1)} \sum_{d \mid r} \varphi_t\left(\frac{r}{d}\right) = r^{k(s-1)+t},$$

which can be seen by applying the Möbius inversion formula to (10), and so from (9),

$$\Gamma_k^{(ks)}(n, r) = r^{k(s-1)+t} \sum_{d \mid r} \frac{c_k(n, d)}{d^t}.$$

On the other hand,

$$\Gamma_k^{(ks)}(n, r) = r^{k(s-1)} \sum_{d^k \mid (n, r^k)_k} d^k \varphi_t\left(\frac{r}{d}\right).$$

If we now set $n = 0$ we obtain the identity

$$\sum_{d \mid r} d^k \varphi_t\left(\frac{r}{d}\right) = \sum_{d \mid r} d^t \varphi_k\left(\frac{r}{d}\right)$$

(see [7], Corollary 5.1).

Example 4. Let $\nu_t(r) = 1$ or 0 according as r is or is not a perfect t-th power. For any complex number s let $\varrho_{s,t}(r)$ be the sum of the s-th powers of all the divisors d of r such that $\frac{r}{d}$ is a perfect t-th power, that is

$$\varrho_{s,t}(r) = \sum_{d \mid r} d^s \nu_t\left(\frac{r}{d}\right).$$

This function was introduced by Gegenbauer. He obtained several arithmetical identities involving $\varrho_{s,t}(r)$, [8], p. 298, these by making use of the zeta function. If we set $g(r) = \nu_t(r)$ for all r, we have immediately that $G_s(r) = \varrho_{s,t}(r)$ and $\Gamma_k^{(s)}(0, r) = \varrho_{s,t}(r)$. Then, from (9), with $ks = k + q$ and $n = 0$, we have

$$\sum_{d \mid r} d^q \varrho_{q,t}\left(\frac{r}{d}\right) \varphi_k(d) = \varrho_{k+q,t}(r),$$

one of Gegenbauer's identies.

4. The Anderson-Apostol theorem and an inversion formula.

We shall begin this section by giving a proof of a generalization of a theorem of Anderson and Apostol ([1], Theorem 2). A function $h(r)$ is said to be factorable if $h(1) = 1$ and $h(r_1 r_2) = h(r_1)\, h(r_2)$ whenever $(r_1, r_2) = 1$ and is said to be completely factorable if $h(1) = 1$ and $h(r_1 r_2) = h(r_1)\, h(r_2)$ always. Let $h(r)$ and $\psi(r)$ be two functions of the positive integral variable r with values in K, and set

$$f(n, r) = \sum_{d^k \mid (n, r^k)_k} h(d)\, \psi\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right), \quad F(r) = f(0, r).$$

Theorem 5. *If $\psi(r)$ is factorable, if $h(r)$ is completely factorable, if and $h(p) \neq 0$, $h(p) \neq \psi(p)$ for all primes p, then*

$$f(n, r) = \frac{F(r)\, \mu(m)\, \psi(m)}{F(m)} \qquad \left(m^k = \frac{r^k}{(n, r^k)_k}\right). \tag{12}$$

Proof. Suppose that $(n_1, n_2) = (n_1, r_2^k)_k = (n_2, r_1^k)_k = (r_1, r_2) = 1$. Then $(n_1, r_1^k)_k$ and $(n_2, r_2^k)_k$ are relatively prime and $(n_1 n_2, r_1^k r_2^k)_k = (n_1, r_1^k)_k (n_2, r_2^k)_k$. Hence $f(n_1 n_2, r_1 r_2) = f(n_1, r_1) f(n_2, r_2)$. Thus it is necessary to prove (12) only in the case when n and r are powers of the same prime p. Let $r = p^\alpha$ and $(n, r^k)_k = p^{\beta k}$. Then

$$f(n, r) = \sum_{i=0}^{\beta} h(p^i)\, \psi(p^{\alpha-i})\, \mu(p^{\alpha-i}). \tag{13}$$

Now, $m = p^{\alpha-\beta}$ ($\alpha \geqq \beta$ always), and so from (13),

$$F(m) = \sum_{i=0}^{\alpha-\beta} h(p^i)\, \psi(p^{\alpha-\beta-i})\, \mu(p^{\alpha-\beta-i}).$$

We now consider three cases:

(i) $\alpha - \beta \geqq 2$. Then (13) vanishes, as does the right-hand side of (12).

(ii) $\alpha - \beta = 1$. Then, from (13), $f(n, r) = -h(p^\beta)\, \psi(p)$, and the right-hand side of (12) is equal to

$$\frac{-[h(p^\alpha) - h(p^{\alpha-1})\, \psi(p)]\, \psi(p)}{h(p) - \psi(p)} = -h(p^\beta)\, \psi(p).$$

(iii) $\alpha = \beta$. Then $m = 1$ and $F(1) = 1$. Hence, the right-hand side of (12) is equal to $F(r) = h(p^\alpha) - h(p^{\alpha-1})\, \psi(p)$ which is precisely $f(n, r)$.

If we set $h(r) = r^k$ and $\psi(r) = 1$ for all r, we have the

Corollary ([4], Theorem 1).

$$c_k(n, r) = \frac{\varphi_k(r)\, \mu(m)}{\varphi_k(m)} \qquad \left(m^k = \frac{r^k}{(n, r^k)_k}\right).$$

Theorem 6 (Inversion formula). *If $f(n, r)$ belongs to the class E_k and*

$$f(n, r) = \sum_{d \mid r} g(d, r)\, c_k(n, d),$$

then

$$g(m, r) = \frac{1}{r^k} \sum_{d \mid r} f\left(\left(\frac{r}{d}\right)^k, r\right) c_k(n, d) \qquad \left(m^k = \frac{r^k}{(n, r^k)_k}\right). \tag{14}$$

Proof. If we set $g(d, r) = \alpha(d)$, then by Theorem 1,

$$g(m, r) = \frac{1}{r^k} \sum_{d \mid r} f\left(\left(\frac{r}{d}\right)^k, r\right) c_k((n, r^k)_k, d),$$

from which (14) follows.

We now set

$$G_s^*(n, r) = m^s G_s\left(\frac{r}{m}\right) \qquad \left(m^k = \frac{r^k}{(n, r^k)_k}\right),$$

and apply Theorem 6 to (9), replacing s by $s + 1$. This results in

$$G_{ks}^*(n, r) = \frac{1}{r^k} \sum_{d \mid r} \Gamma_k^{(k(s+1))}\left(\left(\frac{r}{d}\right)^k, r\right) c_k(n, d). \tag{15}$$

We can now use (15) to obtain some further arithmetical identities.

Example 5. Let $g(r) = \mu(r)$ for all r. Then

$$\Gamma_E^{(s)}(n, r) = \sum_{d^k \mid (n, r^k)_k} d^s\, \mu\left(\frac{r}{d}\right),$$

and so by Theorem 5, with $h(r) = r^s$ and $\psi(r) = 1$ for all r,

$$\Gamma_k^{(s)}\left(\left(\frac{r}{d}\right)^k, r\right) = \frac{\varphi_s(r)\,\mu(d)}{\varphi_s(d)}.$$

Thus,

$$G_{ks}^*(n, r) = \frac{\varphi_{k(s+1)}(r)}{r^k} \sum_{d \mid r} \frac{\mu(d)}{\varphi_{k(s+1)}(d)} c_k(n, d).$$

For $n = 0$ we then have, with $k(s+1) = t$,

$$\sum_{d \mid r} \frac{\mu(d)\,\varphi_k(d)}{\varphi_t(d)} = \frac{r^k \varphi_{t-k}(r)}{\varphi_t(r)} \tag{16}$$

provided $t - k \neq 0$. For $s = 0$ we have the relation

$$\sum_{d \mid r} \frac{c_k(n, d)}{\varphi_k(d)} \mu(d) = \begin{cases} \dfrac{r^k}{\varphi_k(r)} & \text{if } (n, r^k)_k = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Compare this with [7], Corollary 9.

Example 6. Let $\varphi_k^{(s)}(n, r)$ be the number of solutions of the linear congruence

$$n \equiv x_1 + x_2 + \cdots + x_s \pmod{r^k},$$

where $(x_i, r^k)_k = 1$ for $i = 1, 2, \ldots, s$. Cohen has proved ([4], section 4) that

$$\varphi_k^{(s)}(n, r) = \frac{1}{r^k} \sum_{d \mid r} c_k^s\left(\left(\frac{r}{d}\right)^k, r\right) c_k(n, d). \tag{17}$$

Applying the Corollary to Theorem 5 we have

$$\varphi_k^{(s)}(n, r) = \frac{\varphi_k^s(r)}{r^k} \sum_{d \mid r} \frac{\mu^s(d)}{\varphi_k^s(d)} c_k(n, d).$$

When $s = 2$ we obtain a generalization of Nagell's totient function: $\vartheta_k(n, r) = \varphi_k^{(2)}(n, r)$ is the number of integers a such that $1 \leqq a \leqq r^k$ and $(a, r^k)_k = (n - a, r^k)_k = 1$. We thus have

$$\vartheta_k(n, r) = \frac{\varphi_k^2(r)}{r^k} \sum_{d \mid r} \frac{\mu^2(d)}{\varphi_k^2(d)} c_k(n, r).$$

If we apply Theorem 6 to (17) we obtain

$$c_k^s(n, r) = \sum_{d \mid r} \varphi_k^{(s)}\left(\left(\frac{r}{d}\right)^k, r\right) c_k(n, d),$$

and with $n = 0$ and $s = 2$,

$$\sum_{d \mid r} \vartheta_k\left(\left(\frac{r}{d}\right)^k, r\right) \varphi_k(d) = \varphi_k^2(r).$$

5. Some additional identities.

In this section we shall indicate how the identities of [7], section 6 can be generalized by using the results of the last section. Let $g(r) = K(r)\,\mu(r)$, where $K(r)$ is factorable and $K(p) \neq p^s$ for all primes p (in the examples below, the s in this inequality will be replaced by an appropriate value). Then

$$G_s(r) = \sum_{d \mid r} d^s K\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right),$$

$$\Gamma_k^{(s)}(n, r) = \sum_{d^k \mid (n, r^k)_k} d^s K\left(\frac{r}{d}\right) \mu\left(\frac{r}{d}\right) :$$

hence, as usual, $\Gamma_k^{(s)}(0, r) = G_s(r)$, and so by Theorem 5, for any divisor d of r,

$$\Gamma_k^{(s)}\left(\left(\frac{r}{d}\right)^k, r\right) = \frac{G_s(r)\, K(d)\, \mu(d)}{G_s(d)}.$$

Now replace s by $k(s+1)$. Then (15) becomes

$$G_{ks}^*(n, r) = \frac{G_{k(s+1)}(r)}{r^k} \sum_{d \mid r} \frac{K(d)\, \mu(d)}{G_{k(s+1)}(d)}\, c_k(n, d). \tag{18}$$

Example 7. Let t be any complex number not equal to k and let

$$K(r) = r^{k(s+1)} \frac{\varphi_k(r)}{\varphi_t(r)}$$

(then $K(p) \neq p^{k(s+1)}$ for all primes p). We then have

$$G_{k(s+1)}(r) = r^{k(s+1)} \sum_{d \mid r} \frac{\varphi_k(d)\, \mu(d)}{\varphi_t(d)} = \frac{r^{k(s+2)}\, \varphi_{t-k}(r)}{\varphi_t(r)}.$$

Here we have used (16). If we now set $n = 0$, so that $G_{ks}^*(n, r) = G_{ks}(r)$, we have from (18),

$$\varphi_t(r) \sum_{d \mid r} \frac{d^k \varphi_k(d)\, \mu(d)}{\varphi_t(d)} = r^k \varphi_{t-k}(r) \sum_{d \mid r} \frac{\mu(d)\, \varphi_k^2(d)}{d^k \varphi_{t-k}(d)}$$

(see [7], Corollary 14. 3).

Example 8. For $s \neq 0$ we follow Cohen and set

$$Q_s(r) = \sum_{d \mid r} d^s \mid \mu(d) \mid.$$

Now let t be any non-zero complex number and set

$$K(r) = r^{k(s+1)+t} \mu(r).$$

Then $G_{k(s+1)}(r) = r^{k(s+1)} Q_t(r)$ and for $n = 0$ we have $G_{ks}^*(n, r) = G_{ks}(r) = r^{ks} Q_{t+k}(r)$. Hence, from (18),

$$\sum_{d \mid r} \frac{d^t \mid \mu(d) \mid \Phi_k(d)}{Q_t(d)} = \frac{Q_{t+k}(r)}{Q_t(r)}$$

(see [7], Corollary 16. 1).

Example 9. Let $K(r) = r^t$, where $t \neq k(s+1)$ and $t \neq ks$. Then $G_{k(s+1)}(r) = r^t \varphi_{k(s+1)-t}(r)$ and when $n = 0$, $G_{ks}^*(n, r) = G_{ks}(r) = r^t \varphi_{ks-t}(r)$. Then from (18), with $ks = q$, we have

$$\sum_{d \mid r} \frac{\mu(d)\, \varphi_k(d)}{\varphi_{q+k-t}(d)} = \frac{r^{t+k} \varphi_{q-t}(r)}{\varphi_{q+k-t}(r)}.$$

This identity generalizes (16). (16) results when we let $t = 0$.

References.

[1] *R. Douglas Anderson* and *T. M. Apostol*, The evaluation of Ramanujan's sum and generalizations, Duke Math. J. **20** (1953), 211—216.

[2] *Eckford Cohen*, An extension of Ramanujan's sum, Duke Math. J. **16** (1949), 85—90.

[3] *Eckford Cohen*, An extension of Ramanujan's sum II. Additive properties, Duke Math. J. **22** (1955), 543 to 550.

[4] *Eckford Cohen*, An extension of Ramanujan's sum. III. Connection with totient functions, Duke Math. J. **23** (1956), 623—630.

[5] *Eckford Cohen*, A class of arithmetical functions, Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 939—944.

[6] *Eckford Cohen*, Some totient functions, Duke Math. J. **23** (1956), 515—522.

[7] *Eckford Cohen*, Representation of even functions (mod r), I. Arithmetical identities, Duke Math. J. **25** (1958) 401—422.

[8] *L. E. Dickson*, History of the Theory of Numbers. I, Washington 1919.

Eingegangen 20. April 1959.

Stability of perturbed periodic motions [1]).

By *John A. Nohel*, Atlanta, Georgia, U. S. A.

Introduction.

Let x and g be real vectors with n components x_i, g_i $(i = 1, \ldots, n)$ and let t and μ be real. Consider first the nonlinear system

$$x' = g(x, t, \mu), \quad \left(' = \frac{d}{dt}\right), \tag{0. 1}$$

where g and $\frac{\partial g}{\partial x_i}$ are continuous in (x, t, μ) for $|\mu|$ small and for (x, t) in some region V. Let g_x be the matrix with element $\frac{\partial g_i}{\partial x_j}$ in the ith row and j^{th} column. A situation of particular interest is when g has period T in t, and when the unperturbed system

$$x' = g(x, t, 0) \tag{0. 2}$$

has a periodic solution $x = p(t)$ with period T. It is assumed that the region V contains the curve $(p(t), t)$. Questions of existence and stability of periodic solutions $x = q(t, \mu)$ of (0. 1) of period T in t for $|\mu|$ small can, in some cases, be answered easily with the aid of the system of first variation of (0. 1) for $\mu = 0$ with respect to the solution p. This system is

$$y' = g_x(p(t), t, 0)\, y \tag{0. 3}$$

which in this case has periodic coefficients with period T.

The following result is well known. (See, e. g., Coddington and Levinson [1, Ch. 14] or Lefschetz [2, Ch. VIII].)

A. If the first variation (0. 3) has no solution of period T, then for $|\mu|$ small equation (0. 1) has a solution $x = q(t, \mu)$, periodic in t with period T, continuous in (t, μ) and satisfying $q(t, 0) = p(t)$. There is only one such solution for each μ. Moreover, if all the characteristic exponents of (0. 3) have negative real parts [2]), the above conclusion holds and the periodic solution $q(t, \mu)$ is asymptotically stable in the sense of Liapounov provided $|\mu|$ is sufficiently small.

In case (0. 1) is autonomous (i. e., g does not depend on t explicitly) the corresponding system of first variation (0. 3) has $y = p'$ as a solution of period T, and the hypothesis of paragraph A above is never fulfilled. In this case the classical theory is modified, [1] [2].

[1]) This work contains in part results of the author's dissertation written at Massachusetts Institute of Technology in 1953 under the guidance of Professor Norman Levinson, to whom the author expresses his sincere appreciation. This paper was prepared at the Mathematics Research Center, U. S. Army, sponsored by the U. S. Army under contract No. DA-11-022-ORD-2059.

[2]) Equivalently the characteristic multipliers of (0. 3) have magnitude less than 1.

B. If

$$x' = g(x, 0) \tag{0.4}$$

has a periodic solution $x = p(t)$ of period T_0, and if the first variation (0. 3) has no more than one independent solution of period T_0, then

$$x' = g(x, \mu) \tag{0.5}$$

has a periodic solution $x = q(t, \mu)$ for small $|\mu|$, continuous in (t, μ); q has period $T(\mu)$ continuous in μ, and $q(t, 0) = p(t)$, $T(0) = T_0$. Moreover, if $n - 1$ characteristic exponents of (0. 3) have negative real parts, the above conclusion holds and the periodic solution $q(t, \mu)$ has asymptotic orbital stability for small $|\mu|$.

It should be remarked that in A and B stability is an easy consequence of continuity with respect to μ of a certain matrix.

It is easy to give examples of systems of the form (0. 1) and (0. 5) to which the above theory cannot be applied. Such systems arise frequently in nonlinear mechanics. In fact consider the quasilinear system

$$x' = Ax + \mu f(x, t, \mu), \tag{0.6}$$

where f satisfies the same conditions as g did above, and where A is a real constant $n \times n$ matrix having at least one characteristic root of the form iN, N an integer which may be zero. The unperturbed linear system

$$x' = Ax \tag{0.7}$$

now has a periodic solution $p(t)$ with period 2π in t. Moreover, (0. 7) is also the first variation of (0. 6) for $\mu = 0$ with respect to the solution $p(t)$; thus A cannot be applied. For the autonomous system

$$x' = Ax + \mu f(x, \mu) \tag{0.8}$$

with A as above, B cannot be applied since (0. 7) may have more than one independent periodic solution.

Recently Coddington and Levinson, [3] and [1, Ch. 14], gave sufficient conditions for the existence of periodic solutions for systems (0. 6) and (0. 8) for $\mu \neq 0$ and $|\mu|$ sufficiently small under the circumstances just described provided A is in a certain real canonical form. The purpose of this paper is to give further sufficient conditions for asymptotic stability in the sense of Liapounov in case of periodic solutions of (0. 6) and for asymptotic orbital stability for periodic solutions of (0. 8). As in [3] and [1] our conditions only involve the knowledge of solutions of the linear system (0. 7). As is to be expected, for stability μ has to have a fixed sign.

In Section 1 stability criteria are derived for periodic solutions of (0. 6) and (0. 8) in case the matrix A has exactly one pair of characteristic roots i and $-i$, while the remaining eigenvalues of A have negative real parts. This situation is sufficiently simple to make our method clear. The possibility of instability is also considered. Section 2 treats a generalization to the case where A may also have characteristic roots of the form iN, where N is an integer which may be zero. In this connection it is first assumed that the elementary divisors of the characteristic roots iN are simple. The removal of this stringent requirement is then considered. However, this — Theorem 2. 2 — requires an additional hypothesis, not previously made. As a final generalization we consider the system (0. 6) where the constant matrix A is replaced by a real continuous periodic matrix $A(t)$ having period π, and itisa ssumed that the associated unperturbed system has a periodic solution

with period 2π. It is first shown how easily the theory of [3] can be extended to such systems to yield existence of periodic solutions with period 2π in t for $|\mu|$ small, and also how stability criteria for such solutions are obtained.

I. I. Blekhman [4], [5] considered a similar problem independently and by different methods. There the systems (0. 6) and (0. 8) are diagonal, since the assumption about elementary divisors mentioned above is made about all the eigenvalues of A. In addition f is assumed analytic, while here the continuity of certain second partial derivatives of f is sufficient. Blekhman does, however, admit $i\eta$, $\eta \neq$ integer or zero, as an eigenvalue of A; we shall not go into the details of this case in this paper. However, our criteria can be adjusted to include this possibility.

Finally, it should be noted that H. R. Bailey and R. A. Gambill [6] have extended the method developed by Cesari, Hale and Gambill and have given criteria for the stability of periodic solutions of quasilinear second order systems, where the unperturbed system is a *diagonal* system of second order linear differential equations having a periodic solution. Such systems can, of course, be written in the form (0. 6) or (0. 8).

§ 1. a. Nonautonomous Case.

Let A be a real constant $n \times n$ matrix having simple characteristic roots i and $-i$, and suppose that the remaining characteristic roots of A, not necessarily simple, have nonzero real parts. Let x and f be real vectors with n components, f with period 2π in t, and let t and μ be real. Precise smoothness conditions on f in addition to those stated in the Introduction will be given later. Consider the quasilinear system

$$x' = Ax + \mu f(x, t, \mu), \tag{1. 1}$$

and the associated unperturbed linear system

$$x' = Ax, \tag{1. 2}$$

which has a periodic solution $x = p(t)$ with period 2π. Suppose further that (1. 1) has a periodic solution $x = q(t, \mu)$ with period 2π in t, continuous in (t, μ) for $|\mu|$ small and satisfying the condition $q(t, 0) = p(t)$. Such a solution exists if e. g. the conditions given in [3] and [1] are fulfilled. We wish to give sufficient conditions for the asymptotic stability in the sense of Liapounov of the periodic solution $x = q(t, \mu)$.

The change of variable $x = Py$, where P is a suitably chosen real, nonsingular matrix, replaces (1. 1) by a system for y with coefficient matrix $B = P^{-1}AP$, when $\mu = 0$. Moreover, the new system satisfies the same assumptions as (1. 1). It can therefore be assumed without loss of generality that the matrix A in (1. 1) and (1. 2) already has the real canonical form

$$A = \begin{pmatrix} 0 & -1 & \\ 1 & 0 & \\ & & A_0 \end{pmatrix}, \tag{1. 3}$$

where A_0 is a real $(n-2)$ by $(n-2)$ matrix, not necessarily in canonical form, and where the blank parts in (1. 3) are filled in by zeros. Note that the fundamental matrix of (1. 2) which is the identity at $t = 0$ is

$$e^{tA} = \begin{pmatrix} \cos t & -\sin t & \\ \sin t & \cos t & \\ & & e^{tA_0} \end{pmatrix}, \tag{1. 4}$$

and the periodic solution p of (1. 2) is given by $p(t) = e^{tA} a$ where a is a suitably chosen initial vector. We remark in passing that the problem of existence of periodic solutions of (1. 1) for $\mu \neq 0$ rests chiefly in finding a suitable initial vector a such that the initial vector $c(\mu)$ of the periodic solution $x = q(t, \mu)$ of (1. 1) reduces to a when $\mu = 0$. This problem is solved in [3] and [1].

To investigate the stability of the solution q consider the system of first variation of (1. 1) with respect to q,

$$y' = Ay + \mu f_x[q(t, \mu), t, \mu]\, y; \tag{1. 5}$$

this is a linear system with periodic coefficients of period 2π in t. According to the well known theory for such systems a fundamental matrix of (1. 5) is given by

$$Y(t, \mu) = R(t, \mu)\, e^{tS}, \tag{1. 6}$$

where $R(t, \mu)$ is a nonsingular matrix of period 2π in t and S is a n by n matrix independent of t. Since also

$$Y(t + 2\pi, \mu) = Y(t, \mu)\, e^{2\pi S}, \tag{1. 7}$$

it follows that if $Y(t, \mu)$ is the fundamental matrix of (1. 5) which is the identity when $t = 0$, then

$$Y(2\pi, \mu) = e^{2\pi S}. \tag{1. 8}$$

By a well known theorem of Liapounov, [1] and [2], the solution $x = q(t, \mu)$ will be asymptotically stable if the characteristic exponents of (1. 5), i. e. the characteristic roots ϱ_i of the matrix S, have negative real parts, or equivalently, if the characteristic multipliers of (1. 5), i. e. the characteristic roots λ_i of the matrix $Y(2\pi, \mu)$, have magnitude less than 1.

To formulate our sufficient conditions for stability, we study (1. 8) in detail. The fundamental matrix of (1. 5) which is the identity at $t = 0$ satisfies, by the variation of constants, the matrix integral equation

$$Y(t, \mu) = e^{tA} + \mu \int_0^t e^{(t-\tau)A} f_x[q(\tau, \mu), \tau, \mu]\, Y(\tau, \mu)\, d\tau. \tag{1. 9}$$

Thus using (1. 8) one obtains

$$e^{2\pi S} = e^{2\pi A}\left[E + \mu \int_0^{2\pi} e^{-sA} f_x[q(s, \mu), s, \mu]\, Y(s, u)\, ds\right], \tag{1. 10}$$

where E is the identity matrix. In order to estimate the characteristic roots of $e^{2\pi S}$, iterate once by using (1. 9) in (1. 10). This yields

$$\begin{aligned} e^{2\pi S} = e^{2\pi A}\Big\{E &+ \mu \int_0^{2\pi} e^{-sA} f_x[q(s, \mu), s, \mu]\, e^{sA}\, ds \\ &+ \mu^2 \int_0^{2\pi} e^{-sA} f_x[q(s, \mu), s, \mu]\, e^{sA} \int_0^s e^{-\tau A} f_x[q(\tau, \mu), \tau, \mu]\, Y(\tau, \mu)\, d\tau\, ds\Big\}. \end{aligned} \tag{1. 11}$$

Since the right hand side of (1. 11) depends on the solution of the nonlinear problem (1. 1) and since it is clearly desirable to formulate sufficient conditions for stability in terms of solutions of the linear problem (1. 2), a further refinement in (1. 11) is necessary. Assume that $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and $\dfrac{\partial^2 f}{\partial x_i \partial \mu}$ are continuous in (x, t, μ) for $|\mu|$ small and (x, t) in V. Since the $\dfrac{\partial f_i}{\partial \mu}$, $i = 1, \ldots, n$ are continuous, the solution q is differentiable with respect to μ for $|\mu|$ small enough. For a fixed $\mu > 0$ there exists a $\tilde{\mu}$, $0 < \tilde{\mu} < \mu$, such that

$$q(t, \mu) = p(t) + \mu q_\mu(t, \tilde{\mu}), \tag{1. 12}$$

where q_μ is the vector with components $\frac{\partial q_i}{\partial \mu}$, $i = 1, \ldots, n$. Using (1. 12) and the mean value theorem, there exist numbers θ and $\hat{\mu}$, $0 < \theta < 1$, $0 < \hat{\mu} < \mu$, such that

$$(1.13)\qquad \frac{\partial f_i}{\partial x_j}[q(t,\mu), t, \mu] = \frac{\partial f_i}{\partial x_j}[p(t), t, 0] + \mu \frac{\partial^2 f_i}{\partial \mu \partial x_j}[p(t) + \mu q_\mu(t, \tilde{\mu}), t, \hat{\mu}] + \mu \sum_{k=1}^{n} \frac{\partial^2 f_i}{\partial x_k \partial x_j}[p(t) + \theta \mu q(t, \tilde{\mu}), t, \mu] \frac{\partial q_k}{\partial \mu}(t, \tilde{\mu}).$$

Substituting (1. 13) into (1. 11) there results

$$(1.14)\qquad e^{2\pi S} = e^{2\pi A}\{E + \mu D + \mu^2 G(\mu)\},$$

where

$$(1.15)\qquad D = \int_0^{2\pi} e^{-sA} f_x[p(s), s, 0]\, e^{sA}\, ds$$

is a real constant n by n matrix with elements d_{ij} independent of μ, and depending only on the easily found solution of the linear system (1. 2), and where $G(\mu)$ represents a real continuous n by n matrix which satisfies

$$(1.16)\qquad G(\mu) = O(1),\ \mu \to 0.$$

Since from (1. 4)

$$(1.17)\qquad e^{2\pi A} = \begin{pmatrix} 1 & 0 & \\ 0 & 1 & \\ & & e^{2\pi A_0} \end{pmatrix},$$

one finds easily that $e^{2\pi S}$ has the form

$$(1.18)\qquad e^{2\pi S} = \begin{pmatrix} 1 + \mu d_{11} + O(\mu^2) & \mu d_{12} + O(\mu^2) & O(\mu) \\ \mu d_{21} + O(\mu^2) & 1 + \mu d_{22} + O(\mu^2) & \\ O(\mu) & & C \end{pmatrix},$$

where

$$(1.19)\qquad C = e^{2\pi A_0} + O(\mu),$$

and where the other terms not explicitly written out are of order $O(\mu)$. The 2 by 2 submatrix

$$(1.20)\qquad D_2 = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}$$

of D will play an important part below. Note also that the 2 by 2 submatrix in the upper left hand corner of the matrix $e^{2\pi S}$ in (1. 18) is simply $E_2 + \mu D_2 + O(\mu^2)$, where E_2 is the 2 by 2 unit matrix.

Since the conditions of Coddington and Levinson, [3] and [1], for the existence of a unique periodic solution $q(t, \mu)$ of (1. 1) of period 2π in t for $|\mu|$ sufficiently small and with $q(t, 0) = p(t)$ are not needed in our proof they will not be restated here. It should be remarked, however, that we merely require the solution q to exist and have the above properties, and we do not ask that all conditions given in [3] and [1] are fulfilled.

Theorem 1. 1. *Assume that*

(H 1) $\mu > 0$,

(H 2) $\frac{\partial^2 f}{\partial x_i \partial x_j}$ *and* $\frac{\partial^2 f}{\partial x_i \partial \mu}$ $i, j = 1, \ldots, n$, *are continuous in* (x, t, μ) *for* (x, t) *in a vicinity* V *containing the periodic solution* $p(t) = e^{tA} a$ *of* (1. 2) *with period* 2π *in its interior, and for* $0 \leqq \mu < \eta$ *for some* $\eta > 0$,

(H 3) *The submatrix D_2 defined by* (1. 20) *has both of its characteristic roots with negative real parts,*

(H 4) *The $n-2$ by $n-2$ submatrix A_0 of A has all its characteristic roots with negative real parts.*

Then for $\mu > 0$ and μ sufficiently small the solution $x = q(t, \mu)$ of (1. 1), *A in the canonical form* (1. 3), *having period 2π in t and satisfying $q(t, 0) = p(t)$ is asymptotically stable.*

Remark. If $\mu < 0$ assumption (H 3) about the characteristic roots of D_2 would have to be reversed.

Proof of Theorem 1. 1: It will be shown that the characteristic roots $\lambda_i(\mu)$ of the matrix $e^{2\pi S}$ have the property $|\lambda_i(\mu)| < 1$, $i = 1, \ldots, n$. This, as pointed out above, implies that the characteristic exponents $\varrho_i(\mu)$, $i = 1, \ldots, n$, of the system of first variation (1. 5) all have negative real parts. The conclusion then follows from Liapounov's well known theorem, [1], [2]. Note that assumption (H 2) is needed to justify (1. 13), (1. 14), (1. 15) and (1. 16).

Consider now the characteristic polynomial $f(\lambda, \mu)$ of the matrix $e^{2\pi S}$. From (1. 18) one obtains

$$(1.21)\quad f(\lambda,\mu) = \det[\lambda E - e^{2\pi S}] = \left|\begin{array}{cc|c} (\lambda-1)-\mu d_{11}+O(\mu^2) & -\mu d_{12}+O(\mu^2) & \\ -\mu d_{21}+O(\mu^2) & (\lambda-1)-\mu d_{22}+O(\mu^2) & O(\mu) \\ \hline & O(\mu) & \lambda E_{n-2}-C \end{array}\right|$$

where E_{n-2} is the $n-2$ by $n-2$ identity matrix. Now expand the determinant (1. 21) by Laplace's development using the first two rows. This gives

$$(1.22)\quad f(\lambda,\mu) = \det[(\lambda-1)E_2 - \mu D_2]\det[\lambda E_{n-2} - C] + O[(\lambda-1)\mu^2] + O(\mu^3).$$

Defining

$$(1.23)\quad g(\lambda,\mu) = \det[(\lambda-1)E_2 - \mu D_2]\det[\lambda E_{n-2} - e^{2\pi A_0}],$$

and making use of (1. 19), there results

$$(1.24)\quad f(\lambda,\mu) = g(\lambda,\mu) + O[(\lambda-1)^2\mu] + O[(\lambda-1)\mu^2] + O[\mu^3].$$

As $\mu \to 0$, $n-2$ zeros, $\lambda_i(\mu)$, $i = 3, 4, \ldots, n$, of $f(\lambda, \mu)$ approach the zeros of $\det[\lambda E_{n-2} - e^{2\pi A_0}]$. Let these latter zeros be denoted by λ_i', $i = 3, 4, \ldots, n$. By assumption (H 4), $|\lambda_i'| < 1$, $i = 3, 4, \ldots, n$. Since $f(\lambda, \mu)$ is a polynomial in λ of degree n with leading coefficient one and with the remaining coefficients continuous functions of μ, its zeros are continuous functions of μ. Hence by continuity of $\lambda_i(\mu)$ it follows that for $|\mu|$ sufficiently small $|\lambda_i(\mu)| < 1$, $i = 3, 4, \ldots, n$. Note that up to this point assumption (H 1) has not been essential.

As $\mu \to 0$ two zeros of $f(\lambda, \mu)$, $\lambda_i(\mu)$, $i = 1, 2$, approach 1. To these the above argument cannot be applied, and use will now be made of assumptions (H 3) and (H 1). Before doing this it should be noted that (H 3) is equivalent to the requirement that $\operatorname{tr} D_2 < 0$ and $\det D_2 > 0$, where $\operatorname{tr} D_2$ is the trace of the matrix D_2.

Let δ_i, $i = 1, 2$, be the characteristic roots of the matrix D_2, and let $\lambda_i'(\mu)$, $i = 1, 2$, be the two zeros of $g(\lambda, \mu)$ which reduce to 1 when $\mu = 0$; these, of course, are the zeros of $\det[(\lambda-1)E_2 - \mu D_2]$. Thus

$$(1.25)\quad \lambda_i'(\mu) = 1 + \mu\delta_i,\ i = 1, 2.$$

Note that δ_i, $i = 1, 2$, are independent of μ. By (H 1) and (H 3) it is clear that $|\lambda_i'(\mu)| < 1$, $i = 1, 2$, if $\mu > 0$ is sufficiently small. To complete the proof it must be shown that the zeros $\lambda_i(\mu)$, $i = 1, 2$, of $f(\lambda, \mu)$ corresponding to the zeros $\lambda_i'(\mu)$, $i = 1, 2$, of $g(\lambda, \mu)$ also have this property provided $\mu > 0$ is sufficiently small. It is necessary to distinguish two cases.

Case 1. $\delta_1 \neq \delta_2$. Let Γ be the boundary of a circle in the complex λ plane, with center at $\lambda_1'(\mu)$, given by

$$\lambda = \lambda_1' + re^{i\theta}, \quad 0 \leqq \theta \leqq 2\pi, \tag{1.26}$$

where $|\lambda_i'| < 1$ and where $r = \varepsilon\mu$, with ε a fixed positive number satisfying the inequality

$$\varepsilon < \tfrac{1}{2}\,|\delta_1 - \delta_2|. \tag{1.27}$$

Then for $\mu > 0$ small enough Γ lies entirely inside the unit circle in the λ plane, and Γ contains no zero of $g(\lambda, \mu)$ in its interior except λ_1'. Moreover it is easy to show with the aid of (1. 23), (1. 25) and (1. 26) that on Γ

$$|g(\lambda, \mu)| > \frac{\varepsilon^2\mu^2}{2}\,|\det(E_{n-2} - e^{2\pi A_0})|, \tag{1.28}$$

where, by (H 4), $\det(E_{n-2} - e^{2\pi A_0}) \neq 0$.

From (1. 25) and (1. 26) it is further clear that on Γ, the error terms in $f(\lambda, \mu)$ are in fact $O(\mu^3)$. Hence on Γ there exists a constant $k > 0$ such that

$$|f(\lambda, \mu) - g(\lambda, \mu)| \leqq k\mu^3, \tag{1.29}$$

$\mu > 0$ and sufficiently small. Therefore by Rouché's Theorem for $\mu > 0$ and small enough $g(\lambda, \mu)$ and $f(\lambda, \mu) = g(\lambda, \mu) + \big(f(\lambda, \mu) - g(\lambda, \mu)\big)$ have the same number of zeros inside Γ, namely one zero. But since for μ small enough Γ lies entirely inside the unit circle, the zero $\lambda_1(\mu)$ of $f(\lambda, \mu)$ which corresponds to the zero $\lambda_1'(\mu)$ of $g(\lambda, \mu)$ also has the property that $|\lambda_1(\mu)| < 1$. Exactly the same argument applies to the zero λ_2' of $g(\lambda, \mu)$, and thus for $\mu > 0$ sufficiently small we have also that $|\lambda_2(\mu)| < 1$.

Case 2. $\delta_1 = \delta_2 = \delta$. Here the above argument is entirely analogous, except the number ε is chosen to satisfy the inequality

$$\varepsilon < \frac{|\delta|}{2}, \tag{1.30}$$

and Rouché's Theorem yields that $f(\lambda, \mu)$ and $g(\lambda, \mu)$ have the same number of zeros inside Γ — this time two zeros.

Thus it has been shown that for $\mu > 0$ and sufficiently small all zeros $\lambda_i(\mu)$ of the characteristic polynomial $f(\lambda, \mu)$ of the matrix $e^{2\pi S}$ satisfy

$$|\lambda_i(\mu)| < 1, \quad i = 1, \ldots, n, \tag{1.31}$$

and this completes the proof.

A well known result in stability theory for ordinary differential equations, [1] and [2], applied to the present situation, states that if at least one characteristic exponent of the system of first variation (1. 5) has a positive real part, then the periodic solution $x = q(t, \mu)$ of (1. 1) of period 2π in t is not stable. This result combined with the method of proof in Theorem 1. 1 easily yields the following:

Corollary 1. 1. *In assumption* (H 3) *of Theorem* 1. 1 *let the matrix* D_2 *have at least one characteristic root with a positive real part, and let all other hypotheses of Theorem* 1. 1

hold as before. Then for $\mu > 0$ and μ sufficiently small, the solution $x = q(t, \mu)$ of (1. 1) with period 2π in t, satisfying $q(t, 0) = p(t)$, is not stable.

Remarks: 1. Instability can also result if the matrix A_0 in (H 4) has a characteristic root with a positive real part.

2. If D_2 has a characteristic root with zero real part no conclusion can be reached about the stability of the solution $q(t, \mu)$ of (1. 1) for small $|\mu|$ from what has been said so far. It is clear, however, that one could carry out an additional iteration in (1. 11) in order to obtain a more precise estimate of the eigenvalues of the matrix $e^{2\pi S}$.

b. Autonomous Case.

Consider now the quasilinear autonomous system

$$x' = Ax + \mu f(x, \mu), \tag{1. 32}$$

where A is in the canonical form (1. 3). Thus all previous statements, about the linear system (1. 2) hold. The only change here is that f does not depend on t explicitly. Since the matrix $e^{2\pi A}$ does not have 1 as a simple characteristic root, Theorem B stated in the Introduction cannot be applied.

In [3] and [1, Ch. 14] sufficient conditions for the existence of a unique solution $x = q(t, \mu)$ of (1. 32) with period $2\pi + \tau(\mu)$ in t are given, provided $|\mu|$ is sufficiently small. The solution q is continuous in (t, μ) for all t and $|\mu|$ sufficiently small; moreover, $\tau(\mu)$ is a continuous function of μ which satisfies

$$\lim_{\mu \to 0} \frac{\tau(\mu)}{\mu} = \nu_0, \tag{1. 33}$$

where ν_0 is a constant. The solution $q(t, \mu)$ satisfies the condition

$$q(t, 0) = p(t) = e^{tA} a, \tag{1. 34}$$

p a solution of (1. 2) with period 2π, and a a suitably chosen initial vector independent of μ. Sufficient conditions for asymptotic orbital stability of such a solution q will be obtained below. As in part a., the conditions of [3] and [1, Ch. 14] will not be used explicitly.

The system of first variation with respect to the solution q of (1. 32) is

$$y' = Ay + \mu f_x[q(t, \mu), \mu]\, y, \tag{1. 35}$$

a linear system with periodic coefficients with period $2\pi + \tau(\mu)$. The characteristic exponents of (1. 35) can therefore be determined from the relation

$$Y(2\pi + \tau(\mu), \mu) = e^{[2\pi + \tau(\mu)]S}, \tag{1. 36}$$

where $Y(t, \mu)$ is the fundamental matrix of (1. 35) which is the identity at $t = 0$. In this case one characteristic multiplier of (1. 35), say $\lambda_1(\mu) = 1$, since $y = q'$ is a periodic solution of (1. 35) with period $2\pi + \tau(\mu)$ in t. By a well known theorem, [1], [2], on asymptotic orbital stability it will be enough to find sufficient conditions which insure that the remaining multipliers $\lambda_i(\mu)$, $i = 2, \ldots, n$, have magnitude less than 1 for $|\mu|$ sufficiently small. As before it will be clear that μ must have a fixed sign.

By the method of part a., used in equations (1. 9)—(1. 13) inclusive, one obtains

$$(1.37)\qquad e^{[2\pi+\tau(\mu)]S} = e^{[2\pi+\tau(\mu)]A}\{E + \mu\int_0^{2\pi+\tau(\mu)} e^{-sA} f_x[p(s), 0]\, e^{sA} ds + \mu^2(G(\mu)],$$

where $G(\mu)$ is a real continuous n by n matrix satisfying (1. 16). Writing $\int_0^{2\pi+\tau(\mu)} = \int_0^{2\pi} + \int_{2\pi}^{2\pi+\tau(\mu)}$, and using (1. 33), it is easily seen that $\mu\int_{2\pi}^{2\pi+\tau(\mu)} e^{-As} f_x[p(s), 0]\, e^{As} ds$ is also $O(\mu^2)$. Thus letting

$$(1.38)\qquad D = \int_0^{2\pi} e^{-sA} f_x[p(s), 0]\, e^{sA} ds,$$

(1. 37) becomes

$$(1.39)\qquad e^{[2\pi+\tau(\mu)]S} = e^{[2\pi+\tau(\mu)]A}\{E + \mu D + O(\mu^2)\},$$

where the real n by n matrix D depends only on the completely known solution of the linear system (1. 2) and D is independent of μ. From (1. 4) it follows that

$$(1.40)\qquad e^{[2\pi+\tau(\mu)]A} = \begin{pmatrix} \cos\tau(\mu) & -\sin\tau(\mu) & \\ \sin\tau(\mu) & \cos\tau(\mu) & \\ & & e^{[2\pi+\tau(\mu)]A_0} \end{pmatrix},$$

and from (1. 33) it is seen that $\tau(\mu) = \nu_0\mu + o(\mu)$, $|\mu|$ small. Using this and (1. 40) in (1. 39) there results

$$(1.41)\qquad e^{[2\pi+\tau(\mu)]S} = \left(\begin{array}{cc|c} 1 + \mu d_{11} + O(\mu^2) & \mu d_{12} - \nu_0\mu + O\mu^2) & \\ \mu d_{21} + \nu_0\mu + O(\mu^2) & 1 + \mu d_{22} + O(\mu^2) & O(\mu) \\ \hline \multicolumn{2}{c|}{O(\mu)} & C \end{array}\right),$$

where the elements of the $(n-2)$ by $(n-2)$ matrix C differ from the elements of the matrix $e^{2\pi A_0}$ by terms of order μ. Again let D_2 represent the 2 by 2 submatrix of D defined previously by (1. 20) and also define the matrix

$$(1.42)\qquad D_2^* = \begin{pmatrix} d_{11} & d_{12} - \nu_0 \\ d_{21} + \nu_0 & d_{22} \end{pmatrix},$$

and note that $\operatorname{tr} D_2 = \operatorname{tr} D_2^*$.

Theorem 1. 2. *Let assumptions* (H 1), (H 2), *and* (H 4) *in Theorem* 1. 1. *hold. In place of* (H 3) *suppose that:*

(H 3′) *The matrix D_2^* has a characteristic root with negative real part. Then for $\mu > 0$ and sufficiently small the solution $x = q(t, \mu)$ of* (1. 32) *with period $2\pi + \tau(\mu)$ in t, satisfying $q(t, 0) = p(t)$ where p is a solution of* (1. 2) *with period 2π and $\lim_{\mu\to 0}\frac{\tau(\mu)}{\mu} = \nu_0$, has asymptotic orbital stability.*

The *proof of Theorem* 1. 2 is accomplished in much the same way as the proof of Theorem 1. 1 and need not be given. The only difference is that here one knows that the characteristic polynomial $f(\lambda, \mu)$ of the matrix $e^{2\pi S}$ has $\lambda_1(\mu) = 1$ as a root, and by the classical theorem on orbital stability one merely needs to the remaining zeros $\lambda_i(\mu)$ satisfy for $\mu > 0$ and sufficiently small, $|\lambda_i(\mu)| < 1$, $i = 2, 3, \ldots, n$.

Remarks: 1. Assumption (H 3′) is equivalent to requiring that $\operatorname{tr} D_2^* = \operatorname{tr} D_2 < 0$. Thus the constant ν_0 is actually not involved in the conditions for orbital stability. Moreover, it is easy to compute $\operatorname{tr} D_2$ from (1. 38). Using (1. 4) one finds

$$\operatorname{tr} D_2 = \int_0^{2\pi} \left\{ \frac{\partial f_1}{\partial x_1} [p(s), 0] + \frac{\partial f_2}{\partial x_2} [p(s), 0] \right\} ds. \tag{1. 43}$$

Thus (H 3′) merely requires that this integral is negative.

2. The condition found in the above remark is essentially the Poincaré criterion for the orbital stability of a given limit cycle derived in Stoker [7, Appendix V] for systems of two equations without a parameter. In this connection see also Lefschetz [2, pg. 227].

§ 2. Generalizations.

Consider the quasilinear system (1. 1) with μ and f exactly as before. However, let now the real constant matrix A have at least one pair of characteristic roots $\pm iN$, where N is a positive integer which may be zero. The matrix A may also have zero as an eigenvalue of odd multiplicity, and the remaining characteristic roots of A have nonzero real parts. While the eigenvalues $\pm iN$ may be multiple, it will first be assumed that the elementary divisors of all those eigenvalues of A having zero real part are simple. The more general situation, requiring more severe hypotheses on the matrix D below will be considered subsequently.

With these assumptions it is clear that there exists a real nonsingular matrix P such that $P^{-1}AP$ is in the canonical form shown below. Therefore, it will be assumed at the outset that

$$A = \begin{pmatrix} A_1 & & & \\ & \ddots & & \\ & & A_k & \\ & & & 0 \\ & & & & A_\gamma \end{pmatrix}. \tag{2. 1}$$

Each A_j, $j = 1, \ldots, k$ is a matrix of α_j rows and columns (α_j even) of the form

$$A_j = \operatorname{diag}[S_j, \ldots, S_j] \tag{2. 2}$$

with

$$S_j = \begin{pmatrix} 0 & -N_j \\ N_j & 0 \end{pmatrix}, \tag{2. 3}$$

where α_j is the multiplicity of the eigenvalues $\pm iN_j$, N_j integers which may be zero. A_j may have only two rows and columns in which case it is S_j. In (2. 1), 0 is the zero element (to allow for the possibility of zero as an eigenvalue of odd multiplicity). In case zero is not an eigenvalue of A this term is of course omitted from (2. 1). Finally A_γ, which need not be in canonical form, contains all these eigenvalues of A with nonzero real part and has

$$\gamma = n - \left(\sum_{j=1}^{k} \alpha_j + 1 \right) \tag{2. 4}$$

rows and columns. For convenience let $\alpha = \sum_{j=1}^{k} \alpha_j$.

A fundamental matrix of the linear system (1. 2) which is the identity when $t = 0$ is given by

$$(2.5)\qquad e^{tA} = \begin{pmatrix} e^{tA_1} & & & & \\ & \ddots & & & \\ & & e^{tA_k} & & \\ & & & 1 & \\ & & & & e^{tA_\gamma} \end{pmatrix};$$

here

$$(2.6)\qquad e^{tA_j} = \operatorname{diag}\,[e^{tS_j}, \ldots, e^{tS_j}], \quad j = 1, \ldots, k,$$

where

$$(2.7)\qquad e^{tS_j} = \begin{pmatrix} \cos N_j t & -\sin N_j t \\ \sin N_j t & \cos N_j t \end{pmatrix}.$$

It is clear that the unperturbed linear system (1. 2) with A given by (2. 1) again has a periodic solution p with period 2π given by $p(t) = e^{tA}a$ where a is a suitably chosen initial vector. It will be assumed that the nonlinear system (1. 1) has for small $|\mu|$ a periodic solution $x = q(t, \mu)$ continuous in (t, μ), with period 2π in t, and such that $q(t, 0) = p(t)$. Such a solution exists, if for example the conditions of Coddington and Levinson [3] and [1] hold. To obtain sufficient conditions for the asymptotic stability of the solution $x = q(t, \mu)$ of (1. 1) for small $|\mu|$ one proceeds as before via the system of first variation with respect to q, and one finds, as in (1. 5)—(1. 14) inclusive, that the characteristic multipliers of the system of first variation are the characteristic roots of the matrix $e^{2\pi S}$ given by

$$(2.8)\qquad e^{2\pi S} = e^{2\pi A}[E + \mu D + \mu^2 G(\mu)]$$

where D is given by (1. 15) and the continuous matrix $G(\mu)$ satisfies (1. 16).

It follows immediately from (2. 5) that

$$(2.9)\qquad e^{2\pi S} = \left(\begin{array}{cccc|c} 1+\mu d_{11}+O(\mu^2) & \mu d_{12}+O(\mu^2) & \cdots & \mu d_{1,\alpha+1}+O(\mu^2) & \\ \mu d_{21}+O(\mu^2) & 1+\mu d_{22}+O(\mu^2) & \cdots & \mu d_{2,\alpha+1}+O(\mu^2) & \\ \vdots & & \ddots & & O(\mu) \\ \mu d_{\alpha+1,1}+O(\mu^2) & \cdots & & 1+\mu d_{\alpha+1,\alpha+1}+O(\mu^2) & \\ \hline & & O(\mu) & & C_\gamma \end{array}\right),$$

where the elements of the matrix C_γ differ from those of $e^{2\pi A_\gamma}$ by terms of order $O(\mu)$. For convenience let

$$(2.10)\qquad D_{\alpha+1} = (d_{ij}), \quad i, j = 1, \ldots, \alpha + 1.$$

Theorem 2. 1. *Assume that*

(H 1) $\qquad \mu > 0$

(H 2) $\dfrac{\partial^2 f}{\partial x_i \partial x_j}$ *and* $\dfrac{\partial^2 f}{\partial x_i \partial \mu}$, $i, j = 1, \ldots, n$ *are continuous in* (x, t, μ) *for* (x, t) *in a*

vicinity V containing the periodic solution $p(t) = e^{tA} a$ with period 2π of (1. 2) in its interior, and for $0 \leqq \mu < \eta$ for some $\eta > 0$.

(H 3) *The submatrix $D_{\alpha+1}$ of D defined by (2. 10) has all its characteristic roots δ_i, $i = 1, \ldots, \alpha + 1$, with negative real parts.*

(H 4) *The submatrix A_γ of A has all its $\gamma = n - \alpha - 1$ characteristic roots with negative real parts.*

Then for $\mu > 0$ and sufficiently small the solution $x = q(t, \mu)$ of (1. 1), A in the canonical form (2. 1), having period 2π in t and satisfying $q(t, 0) = p(t)$ is asymptotically stable.

Proof: The method of proof is similar to the proof of Theorem 1. 1, and will not be given in detail. By Laplace's development, the characteristic polynomial of the matrix $e^{2\pi S}$, (2. 9), is found to be

$$f(\lambda, \mu) = g(\lambda, \mu) + O((\lambda - 1)^{\alpha+1}\mu) + \cdots + O((\lambda - 1)\mu^{\alpha+1}) + O(\mu^{\alpha+2}), \tag{2. 11}$$

where

$$g(\lambda, \mu) = \det\,[(\lambda - 1) E_{\alpha+1} - \mu D_{\alpha+1}] \det\,[\lambda E_\gamma - e^{2\pi A_\gamma}]; \tag{2. 12}$$

$f(\lambda, 0)$ has one as a zero of multiplicity $\alpha + 1$, while its remaining γ zeros have magnitude less than 1 by (H 4). Assumption (H 3) insures that all n zeros of $g(\lambda, \mu)$ have magnitude less than 1 provided $\mu > 0$ and sufficiently small. The zeros, $\lambda_i(\mu)$, $i = \alpha + 2, \ldots, n$, of $f(\lambda, \mu)$ which approach the zeros of $\det\,[\lambda E_\gamma - e^{2\pi A_\gamma}]$ are handled exactly as before. For the remaining zeros Rouché's theorem is employed again. In this connection we remark only that if $\lambda_i' = 1 + \mu\delta_i$ is a zero of $\det\,[(\lambda - 1) E_{\alpha+1} - \mu D_{\alpha+1}]$, then one lets

$$\varepsilon = \frac{1}{2} \min_{i \neq j} |\delta_i - \delta_j|, \tag{2. 13}$$

and one finds that on Γ, where Γ was defined previously,

$$|g(\lambda, \mu)| > \varepsilon^{\alpha+1} \frac{\mu^{\alpha+1}}{2^{\alpha/2}} |\det\,[E_\gamma - e^{2\pi A_\gamma}]|, \tag{2. 14}$$

and

$$|f(\lambda, \mu) - g(\lambda, \mu)| \leqq k\mu^{\alpha+2}. \tag{2. 15}$$

Rouche's theorem can be applied again, and thus it is shown that $|\lambda_i(\mu)| < 1$, $i = 1, \ldots, n$, for $\mu > 0$ and μ sufficiently small. Applying Liapounov's theorem this completes the proof.

Remarks: 1. A theorem similar to Corollary 1. 1 is easily established.

2. In the autonomous case (1. 32) and A in the canonical form (2. 1) one proceeds as in § 1b. Asymptotic orbital stability can be established if the submatrix $D^*_{\alpha+1}$ analogous to D^*_2 (1. 42) has α characteristic roots with negative real parts, and the other hypotheses as in Theorem 2. 1.

To study what happens if the assumption about the simple elementary divisors of those eigenvalues of A having zero real parts is removed, consider first the following simple example. In the system (1. 1) let

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2. 16}$$

and x and f are vectors with only two components. Then e^{tA} is simply

$$e^{tA} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \tag{2. 17}$$

and the unperturbed linear system (1. 2) has a periodic solution $p(t)$ with period 2π. Moreover, the theory of [3] and [1] is applicable and guarantees, for small $|\mu|$, a solution $q(t, \mu)$ with period 2π in t. Employing the now familiar procedure for studying the stability of this solution, we find that the matrix $e^{2\pi S}$ for the determination of characteristic multipliers of the relevant system of first variation is given by

$$e^{2\pi S} = e^{2\pi A}[E_2 + \mu D + O(\mu^2)], \tag{2. 18}$$

where $e^{2\pi A}$ is found from (2. 17) and D from (1. 15) using (2. 17). When carried out (2. 18) gives

$$e^{2\pi S} = \begin{pmatrix} 1 + \mu d_{11} + O(\mu^2) & \mu d_{12} + O(\mu^2) \\ 2\pi(1 + \mu d_{11}) + \mu d_{21} + O(\mu^2) & 1 + \mu d_{22} + 2\pi\mu d_{12} + O(\mu^2) \end{pmatrix}. \tag{2. 19}$$

Suppose again that both characteristic roots δ_1 and δ_2 of D have negative real parts. A simple calculation shows that the eigenvalues of $e^{2A\pi}[E_2 + \mu D]$ need not have magnitude less than 1 for $\mu > 0$ and μ small, unless $d_{12} = 0$. If $d_{12} = 0$ and if (H 1), (H 2), and (H 3) of Theorem 1. 1 hold, then the conclusion of Theorem 1. 1 holds with A given by (2. 16). This example shows that more severe restrictions are required if the assumptions about elementary divisors made at the beginning of Section 2 do not hold.

To state a more general result of this nature let A in (1. 1) have the following canonical form (in place of (2. 1)):

$$\begin{pmatrix} A_1 & & & & & \\ & \ddots & & & & \\ & & A_k & & & \\ & & & B_1 & & \\ & & & & \ddots & \\ & & & & & B_m \\ & & & & & & A_\gamma \end{pmatrix}. \tag{2. 20}$$

Each A_j, $j = 1, \ldots, k$, is a matrix of α_j rows and columns (α_j even) given by

$$A_j = \begin{pmatrix} S_j & O_2 & \cdots & \cdots & O_2 \\ E_2 & \ddots & \ddots & & \vdots \\ O_2 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & O_2 \\ O_2 & \cdots & O_2 & E_2 & S_j \end{pmatrix} \tag{2. 21}$$

where S_j is given by (2. 3) and O_2 is the two by two zero matrix. A_j may have only two rows and columns in which case it is S_j. Each B_j, $j = 1, \ldots, m$ is a matrix of β_j rows and columns of the form

$$B_j = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & 0 \quad 1 & 0 \end{pmatrix}. \tag{2. 22}$$

Each B_j may have only 1 row and column in which case it is zero. Let

$$\alpha + \beta = \sum_{j=1}^{k} a_j + \sum_{j=1}^{m} \beta_j,$$

and

$$\gamma = n - \alpha - \beta. \tag{2. 23}$$

The matrix A_γ in (2. 20) has γ rows and columns and its eigenvalues have nonzero real parts as before. The system (1. 2) with A in the general canonical form (2. 20) again has a periodic solution $p(t)$ with period 2π and the perturbed system (1. 1) has for small $|\mu|$ a periodic solution $x = q(t, \mu)$ satisfying familiar properties under suitable conditions. The following rather restrictive stability criterion of the solution q is easily established.

Theorem 2. 2. *Let* (H 1), (H 2), (H 4) *of Theorem* 2. 1 *hold. In place of* (H 3) *suppose that* ($\widetilde{\text{H}}$ 3) *the submatrix*

$$D_{\alpha+\beta} = (d_{ij}) \quad i, j = 1, \ldots, \alpha + \beta, \tag{2. 24}$$

of D (1. 15) *has all its eigenvalues with negative real parts and*

$$d_{ij} = 0 \text{ for } j > i; \tag{2. 25}$$

i. e. the submatrix $D_{\alpha+\beta}$ *is lower triangular.*

Then for $\mu > 0$ *and sufficiently small the solution* $x = q(t, \mu)$ *of* (1. 1), A *in the canonical form* (2. 20), *having period* 2π *in* t *and satisfying* $q(t, 0) = p(t)$ *is asymptotically stable.*

Proof. Employing previous notation the characteristic polynomial $f(\lambda, \mu)$ of the relevant matrix $e^{2\pi S}$ is computed. The only significant change is that assumption ($\widetilde{\text{H}}$ 3) is now needed to insure that all zeros of the relevant polynomial $g(\lambda, \mu)$ should have magnitude less than one. The proof can be completed in much the same manner as the previous result with only obvious modifications.

As a final generalization consider the quasilinear system

$$x' = A(t)\, x + \mu f(x, t, \mu) \tag{2. 26}$$

where μ and f are exactly as in (1. 1), but the constant matrix A in (1. 1) is replaced by the real continuous periodic matrix $A(t)$,

$$A(t + \pi) = A(t). \tag{2. 27}$$

By the theory for linear, *real* periodic systems there exists a real nonsingular matrix $Z(t)$ of *period* 2π and a real constant matrix C such that the unperturbed system

$$x' = A(t)\,x \tag{2.28}$$

has

$$\Phi(t) = Z(t)\,e^{tC} \tag{2.29}$$

for a fundamental matrix. It will be assumed that at least one characteristic multiplier of (2. 28) is 1, or equivalently it may be assumed that the matrix C has a characteristic root of the form iN, where N is an integer which may be zero. This implies that the linear system (2. 28) has a periodic solution with period 2π and therefore the classical results of Theorem A in the Introduction regarding existence and stability of a periodic solution with period 2π in t for $|\mu|$ small are not applicable. Note that it may be assumed without loss of generality that C is in the real canonical form.

While the authors in [3] do not consider the question of existence of a periodic solution of period 2π for (2. 27) when $|\mu|$ is small, it is easy to see how their theory can be applied[3]). In fact let

$$x(t) = Z(t)\,w; \tag{2.30}$$

then (2. 26) becomes

$$w' = Cw + \mu Z^{-1}(t)\,f(Z(t)\,w, t, \mu), \tag{2.31}$$

or, more simply

$$w' = Cw + \mu F(w, t, \mu), \tag{2.32}$$

where F has the same properties as f. At this point it should be noted that even if f in (2. 26) does not depend on t explicitly (2. 32) will not be autonomous. Now (2. 32) is exactly of the form (0. 6) and the associated unperturbed system

$$w' = Cw \tag{2.33}$$

has, because of our assumption regarding characteristic multipliers of (2. 27), a periodic solution $p(t)$ with period 2π. Moreover, C is in the real canonical form. Thus the theory of Coddington and Levinson for nonautonomous systems can be applied to yield sufficient conditions for the existence of a periodic solution $w = q(t, \mu)$ of (2. 32) with period 2π in t provided $|\mu|$ is sufficiently small. Since $Z(t)$ is nonsingular and has period 2π this gives at the same time a periodic solution $x = Z(t)\,q(t, \mu)$ of (2. 26) with period 2π in t for $|\mu|$ small.

It is equally easy to see that our appropriate criteria for asymptotic stability can also be applied to such a solution by considering the transformed system (2. 32). The choice of the particular stability theorem to be applied depends on whether the elementary divisors associated with a characteristic exponent iN of the unperturbed periodic system (2. 28) are simple or not. While there is no need to go into details, we remark only that the relevant matrix D which plays an important role in all the stability theorems is now given by

$$D = \int_0^{2\pi} e^{-sC} Z^{-1}(s)\,f_x[p(s), s, 0]\,Z(s)\,e^{sC}\,ds; \tag{2.34}$$

or simply

$$D = \int_0^{2\pi} \Phi^{-1}(s)\,f_x[p(s), s, 0]\,\Phi(s)\,ds, \tag{2.35}$$

---

[3]) The authors imply this by their statement of problem 4, [1, pg. 370].

where $\Phi(t)$ is the fundamental matrix, (2. 29), of (2. 28) which is the identity when $t = 0$. Thus stability criteria can be given entirely in terms of known solutions of the unperturbed linear system.

Added in Proof:

After this manuscript was completed the author learned that Y. Sibuya has been considering similar problems independently. In "Mathematics Research Center Technical Summary Report No. 102", September 1959, University of Wisconsin, he shows that the same stability criteria derived in this paper for periodic solutions of nonautonomous systems can be obtained by a different method. J. K. Hale informed the author recently that he can obtain results similar to ours, and his work will appear in a forthcoming issue of "Contributions to Nonlinear Oscillations", Vol. V.

Bibliography.

[1] *E. A. Coddington* and *N. Levinson*, Theory of Ordinary Differential Equations, 1955.

[2] *Solomon Lefschetz*, Differential Equations: Geometric Theory, New York 1957.

[3] *E. A. Coddington* and *N. Levinson*, Perturbations of Linear Systems with Constant Coefficients Possessing Periodic Solutions, Contrib. to the Theory of Nonlinear Oscillations, II, Annals of Math. Studies, Princeton Univ. Press, **29** (1952). [M. R.: Vol. 14, No. 10, pg. 981].

[4] *I. I. Blekhman*, On the Problem of Stability of Periodic Solutions of Quasilinear Systems with Many Degrees of Freedom, Dokl. Akad. Nauk S.S.S.R. (N. S.) **104** (1955), 809—812 (Russian). [M. R.: Vol. 17, No. 6, pg. 616].

[5] *I. I. Blekhman*, On the Stability of Periodic Solutions of Quasilinear Autonomous Systems with Several Degrees of Freedom, Dokl. Akad. Nauk S.S.S.R. (N. S.) **112** (1957), 183—186 (Russian). [M. R.: Vol. 19, No. 3, pg. 276].

[6] *H. R. Bailey* and *R. A. Gambill*, On the Stability of Periodic Solutions of Weakly Nonlinear Differential Systems, Jour. of Math. and Mechanics, **6** (1957), 655—667. [M. R.: Vol. 19, No. 8, pg. 856].

[7] *J. J. Stoker*, Nonlinear Vibrations, New York 1950.

Mathematics Research Center, U. S. Army, Madison, Wisc., and Georgia Institute of Technology, Atlanta, Ga.

Eingegangen 21. April 1959.

On Absolute Class Fields of Certain Algebraic Number Fields.

By *Taira Honda* in Tokyo.

Introduction.

Let P be an algebraic number field of finite degree and K a cyclic extension of prime degree l over P such that the number of ambigous classes is equal to 1 [1]). Moreover, assume that the absolute class field L of K is cyclic over K. (These notations will be used throughout this paper.) It is the aim of the present paper to investigate the arithmetical properties of the field L, in particular the structure of the absolute ideal class group $\mathfrak{K}_{\mathsf{L}}$ of L. Our main theorem asserts that $\mathfrak{K}_{\mathsf{L}}$ is isomorphic to the l-fold direct product of the absolute class group $\mathfrak{K}_{\Omega}$ of a certain subfield Ω of L, if K/P is ramified, and to the l-fold direct product of a subgroup of index l of $\mathfrak{K}_{\Omega}$, if K/P is unramified.

This result is concerned with the structure of class field towers over algebraic number fields. (For example, see corollary of the main theorem in § 3.) It implies also a very simple relation between the absolute class groups of intermediate fields of the Galois extension L/P, adding a new result to the study of class number relations pursued by Nehrkorn [4], Kuroda [5], Brauer [6] and others.

In § 1 we shall give some preliminaries and determine in particular the structure of the Galois group $\mathfrak{G}$ of L/P. § 2 deals with Abelian groups with an operator domain isomorphic to $\mathfrak{G}$. The proof of the main theorem follows immediately from the results of § 2 and the class field theory as will be shown in § 3. § 4 deals with the unit group of the absolute class field L. Finally some numerical examples will be given in § 5.

The author wishes to express his cordial thanks to Profs. S. Iyanaga, Y. Kawada, and T. Tamagawa for their kind advices and encouragement. He expresses his hearty gratitude also to Prof. T. Tannaka who read the original manuscript carefully and pointed out an error.

§ 1. Preliminaries.

Let k be any algebraic number field of finite degree, and k' a cyclic extension of k. The number of ambigous classes $a_{\mathsf{k}'/\mathsf{k}}$ of k'/k is given by the following formula:

$$a_{\mathsf{k}'/\mathsf{k}} = \frac{h_{\mathsf{k}} \prod_{\mathfrak{p}} e(\mathfrak{p})}{(\mathsf{k}' : \mathsf{k})\,(\varepsilon : N(\Theta))} \tag{1}$$

where

h_{k} = class number of k,

$e(\mathfrak{p})$ = ramification exponent of prime divisor $\mathfrak{p}$. ($\mathfrak{p}$ runs over all the prime divisors of k.)

ε = units in k,

[1]) For an arbitrarily given P, K satisfying these conditions does not always exist. cf. § 3.

Θ = elements of k' whose norms $N(\Theta) = N_{\mathsf{k}'/\mathsf{k}}(\Theta)$, are units in k.

We derive the following two theorems as special cases of (1).

Theorem 1. *If k' is the absolute class field of k, then we have*

$$a_{\mathsf{k}'/\mathsf{k}} = 1\,^{2)}.$$

Theorem 2. *Assume that $(\mathsf{k}' : \mathsf{k}) = l$ is a prime. If k'/k is ramified, $a_{\mathsf{k}'/\mathsf{k}}$ is divisible by h_{k}. But, if k'/k is unramified, we have*

$$a_{\mathsf{k}'/\mathsf{k}} = \frac{h_{\mathsf{k}}}{l}.$$

(In this paper, "k'/k is unramified" means that all the prime divisors in k including infinite prime divisors are unramified in k'.)

Proof. (This theorem has already been proved as "Satz 1" in Moriya [3]. But we shall give here another proof by making use of the theory of norm residue.)

Let e be the number of the ramifying prime divisors of k. By the norm residue theorem, a unit ε in k is the norm of an element in k', if and only if $\left(\frac{\varepsilon,\, \mathsf{k}'/\mathsf{k}}{\mathfrak{p}}\right) = 1$ for all the prime divisors $\mathfrak{p}$ of k. Since $\left(\frac{\varepsilon,\, \mathsf{k}'/\mathsf{k}}{\mathfrak{p}}\right) = 1$ for unramified $\mathfrak{p}$, we have $(\varepsilon : N(\Theta)) = 1$ for $e = 0$ and hence

$$a_{\mathsf{k}'/\mathsf{k}} = \frac{h_{\mathsf{k}}}{l}$$

for an unramified extension k'/k.

Now assume that $e \geqq 1$. Because of the product formula of norm residue symbol, ε is the norm of an element in k', if and only if

$$\left(\frac{\varepsilon,\, \mathsf{k}'/\mathsf{k}}{\mathfrak{p}_i}\right) = 1$$

for $e - 1$ ramifying prime divisors $\mathfrak{p}_1, \ldots, \mathfrak{p}_{e-1}$. Hence the mapping

$$\varepsilon \to \left\{\left(\frac{\varepsilon,\, \mathsf{k}'/\mathsf{k}}{\mathfrak{p}_i}\right)\right\}_{1 \leq i \leq e-1}$$

defines a homomorphism from the unit group of k into an Abelian group $\mathfrak{t}_{e-1}$ of type $\underbrace{(l, \ldots, l)}_{e-1}$ and $(\varepsilon : N(\Theta))$ is the order of the factor group of the unit group of k by the kernel of this homomorphism. As this group is isomorphic to a subgroup of $\mathfrak{t}_{e-1}$, $(\varepsilon : N(\Theta))$ is a divisor of l^{e-1} and $a_{\mathsf{k}'/\mathsf{k}} = \dfrac{h_{\mathsf{k}} \cdot l^e}{l \cdot (\varepsilon : N(\Theta))}$ is a multiple of h_{k}. q. e. d.

Now let P, K, L be fields satisfying the assumptions of the introduction. By the criterion in Hasse [1] II § 5, L is a normal extension of P. Denote by $\mathfrak{G}$, $\mathfrak{S}$, $\bar{\mathfrak{T}}$ the Galois groups of L/P, L/K and K/P respectively. $\mathfrak{S}$ is generated by a substitution σ of order q, where $q = h_{\mathsf{k}}$. As $(l, q) = 1$ [3]), $\mathfrak{G}$ contains a subgroup $\mathfrak{T} = \{\tau\}$ of order l and for the obvious reason we shall identify $\bar{\mathfrak{T}}$ with $\mathfrak{T}$. Thus the group $\mathfrak{G}$ is perfectly determined by the transformation of σ by τ:

$$\tau^{-1}\sigma\tau = \sigma^a, \quad a^l \equiv 1 \pmod{q}.$$

As for the structure of $\mathfrak{G}$, we have

Lemma 1. *The centralizer of $\mathfrak{T}$ coincides with $\mathfrak{T}$ itself.*

---

[2]) cf. Tannaka [7].

[3]) For example see Moriya [2].

Proof. This is an immediate consequence of Artin's reciprocity law. Let σ' be any element of $\mathfrak{S}$ and c' be the corresponding ideal class of K. Then c'^{τ} corresponds to $\tau^{-1}\sigma'\tau$. Therefore, if $\tau^{-1}\sigma'\tau = \sigma'$, we have $c'^{\tau} = c'$ i. e. c' is an ambigous class of K/P. So, by our assumption, c' is the principal class of K and σ' must be the identity. q. e. d.

Now, let us transform lemma 1 into a more convenient form for our purpose.

Lemma 2. $(a^i - 1, q) = 1$ *for* $1 \leqq i \leqq l-1$.

Proof. Let p be any prime divisor of q. If p divided $a^i - 1$ $(1 \leqq i \leqq l-1)$, we should have

$$\tau^{-i}\sigma^{\frac{q}{p}}\tau^i = \sigma^{\frac{q}{p}a^i}$$

and

$$\frac{q}{p} - \frac{q}{p}a^i = q\frac{1-a^i}{p} \equiv 0 \pmod{q}.$$

Thus $\sigma^{\frac{q}{p}}$ would belong to the centralizer of $\mathfrak{T}$ in contradiction to lemma 1. q. e. d.

Let Ω be the subfield of L belonging to the subgroup $\mathfrak{T}$. Denote by χ_{k} the character of $\mathfrak{G}$ induced by the principal character of the subgroup of $\mathfrak{G}$ belonging to an intermediate field k of L/P. The result of this paper has an intimate connection with the following character relation, though we do not need this to prove the main theorem.

Lemma 3. $\chi_{\mathsf{L}} = \chi_K + l(\chi_\Omega - \chi_P)$.

Taking account of lemma 1, it is easily verified that this is a special case of a character relation given in Ishida [8]. (We can prove this lemma also by direct computation using lemma 2 and the definition of induced characters without difficulty.)

§ 2. Abelian groups with the operator domain $\mathfrak{G}$.

Let $\mathfrak{K}$ be a multiplicative Abelian group which has the group $\mathfrak{G}$ as operator domain. The identity of $\mathfrak{K}$ will be denoted by 1. Assume that the identity of $\mathfrak{G}$ operates on $\mathfrak{K}$ as the identity mapping and that for $\varrho_1, \varrho_2 \in \mathfrak{G}$ and for $C \in \mathfrak{K}$

$$C^{\varrho_1\varrho_2} = (C^{\varrho_1})^{\varrho_2}.$$

Let ϱ be any element of order m of $\mathfrak{G}$. For any element C of $\mathfrak{K}$ we shall call $C^{1+\varrho+\cdots+\varrho^{m-1}}$ *ϱ-norm* of C and denote it by $N_\varrho C$. Denote by ${}_\varrho\mathfrak{K}$ the subgroup of $\mathfrak{K}$ of all the elements whose ϱ-norm is 1. We shall call an element of $\mathfrak{K}$, invariant by φ, *φ-ambigous*. Denote by $\mathfrak{K}_\varphi$ the subgroup of $\mathfrak{K}$ of all the φ-ambigous elements.

The following two mutually dual theorems constitute algebraic part of the proof of the main theorem in § 3.

Theorem 3. *If* $\mathfrak{K}_\sigma = \{1\}$, *we have*

$$\bigcap_{i=0}^{l-1} {}_{\sigma^{-i}\tau\sigma^i}\mathfrak{K} = \{1\}.$$

Proof. Let C be in $\bigcap_{i=0}^{l-1} {}_{\sigma^{-i}\tau\sigma^i}\mathfrak{K}$. This means that

$$(2)\quad \begin{cases} C^{1+\tau+\cdots+\tau^{l-1}} = 1 \\ C^{1+\sigma^{-1}\tau\sigma+\cdots+\sigma^{-1}\tau^{l-1}\sigma} = 1 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ C^{1+\sigma^{-(l-1)}\tau\sigma^{l-1}+\cdots+\sigma^{-(l-1)}\tau^{l-1}\sigma^{l-1}} = 1. \end{cases}$$

In order to carry out the computation smoothly we introduce the group ring $\mathfrak{R}$ of $\mathfrak{G}$ over the ring of rational integers, which is also an operator domain of $\mathfrak{K}$. The elements of $\mathfrak{R}$ which map C to 1 form a right ideal $\mathfrak{J}$ in $\mathfrak{R}$. Thus (2) is transformed into

$$\begin{cases} 1+\tau+\cdots+\tau^{l-1}\equiv 0 \\ 1+\sigma^{-1}\tau\sigma+\cdots+\sigma^{-1}\tau^{l-1}\sigma\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{J}) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1+\sigma^{-(l-1)}\tau\sigma^{l-1}+\cdots+\sigma^{-(l-1)}\tau^{l-1}\sigma^{l-1}\equiv 0 \end{cases}$$

or

$$(3)\qquad \begin{cases} 1+\tau+\cdots+\tau^{l-1}\equiv 0 \\ 1+\tau\sigma^{1-a}+\cdots+\tau^{l-1}\sigma^{1-a}\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{J}). \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 1+\tau\sigma^{(1-a)(l-1)}+\cdots+\tau^{l-1}\sigma^{(1-a^{l-1})(l-1)}\equiv 0. \end{cases}$$

Next we eliminate $\tau,\ldots,\tau^{l-1}$ from (3) by multiplying these congruences by suitable polynomials of σ with integral coefficients from the right, and adding them. Then we have

$$(4)\qquad \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & \sigma^{1-a} & \cdots & \sigma^{1-a^{l-1}} \\ \vdots & \vdots & & \vdots \\ 1 & \sigma^{(1-a)(l-1)} & \cdots & \sigma^{(1-a^{l-1})(l-1)} \end{vmatrix} \equiv 0 \qquad (\mathrm{mod}\ \mathfrak{J}),$$

i.e.

$$\prod_{0\leq i<j\leq l-1}(\sigma^{1-a^i}-\sigma^{1-a^j})\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{J}),$$

or

$$\prod_{0\leq i<j\leq l-1}(1-\sigma^{a^j-a^i})\equiv 0 \qquad (\mathrm{mod}\ \mathfrak{J}).$$

That is

$$(5)\qquad C^{\prod_{0\leq i<j\leq l-1}(1-\sigma^{a^j-a^i})}=1.$$

Since $\sigma^{a^j-a^i}=\sigma^{a^i(a^{j-i}-1)}$ $(0\leqq i<j\leqq l-1)$ are generators of $\mathfrak{S}$ because of lemma 2, we obtain from (5)

$$C=1$$

by making use of the assumption of the theorem repeatedly. q. e. d.

Theorem 4. *If* $\mathfrak{K}_\sigma=\{1\}$, *the product of subgroups* $\mathfrak{K}_\tau, \mathfrak{K}_{\sigma^{-1}\tau\sigma},\ldots,\mathfrak{K}_{\sigma^{-(l-1)}\tau\sigma^{l-1}}$ *is a direct product.*

In order to prove this theorem, we need the elementary

Lemma 4. *Let n, b be positive integers and $b>1$. Define rational functions $f_{n,i}(X)$ $(i=1,2,\ldots)$ of an indeterminate X by induction as follows:*

$$f_{n,1}(X)=X^{n-1},$$

$$f_{n,i+1}(X)=\frac{f_{n,i}(X)-f_{n,i}(X^b)}{X-X^{b^i}}.$$

Then $f_{n,i}(X)$ $(i=1,2,\ldots)$ are all polynomials of X with integral coefficients. In particular, we have

$$f_{n,n}(X)=1,$$

$$f_{n,i}(X)=0 \text{ for } i>n.$$

Proof. In fact $f_{n,i}(X)$ can be expressed explicitly[4]):

$$(6) \qquad f_{n,i}(X) = \sum_{\substack{k_1+\cdots+k_i=n-i \\ k_1\geq 0,\ldots,k_i\geq 0}} X^{k_1b^{i-1}+k_2b^{i-2}+\cdots+k_i}.$$

We shall prove (6) by induction with respect to i. For $i = 1$ formula (6) is trivial. Suppose that it is proved for $i = j$. Then

$$f_{n,j}(X) - f_{n,j}(X^b) = \sum_{\substack{k_1+\cdots+k_j=n-j \\ k_1\geq 0,\ldots,k_j\geq 0}} X^{k_1b^{j-1}+\cdots+k_j} - \sum_{\substack{k_1+\cdots+k_j=n-j \\ k_1\geq 0,\ldots,k_j\geq 0}} X^{k_1b^j+\cdots+k_jb}$$

$$= \sum_{\substack{k_1+\cdots+k_{j+1}=n-j \\ k_1\geq 0,\ldots,k_j\geq 0,k_{j+1}>0}} X^{k_1b^{j-1}+\cdots+k_jb+k_{j+1}} - \sum_{\substack{k_1+\cdots+k_j=n-j \\ k_1>0,k_2\geq 0,\ldots,k_j\geq 0}} X^{k_1b^j+k_2b^{j-1}+\cdots+k_jb}$$

$$= \sum_{\substack{k_1+\cdots+k_{j+1}=n-j \\ k_1\geq 0,\ldots,k_j\geq 0,k_{j+1}>0}} X^{k_1b^j+\cdots+k_{j+1}} - \sum_{\substack{k_1+\cdots+k_{j+1}=n-j \\ k_1>0,k_2\geq 0,\ldots,k_{j+1}\geq 0}} X^{k_1b^j+\cdots+k_{j+1}}$$

$$= (X - X^{b^j}) \sum_{\substack{k_1+\cdots+k_{j+1}=n-j-1 \\ k_1\geq 0,\ldots,k_{j+1}\geq 0}} X^{k_1b^j+\cdots+k_{j+1}}.$$

Hence (6) is true for $i = j + 1$. All the assertions of the lemma follow directly from (6). q. e. d.

Proof of theorem 4. In fact we shall prove that, if $C_1, \ldots, C_i (1 \leq i \leq l)$ are elements of $\mathfrak{K}_\tau$ such that

$$(7) \qquad C_1 C_2^\sigma \cdots C_i^{\sigma^{i-1}} = 1,$$

C_l must be the identity. With the notation of lemma 4 we obtain instead of (7) by putting $b = a$

$$C_1^{f_{1,1}(\sigma)} \cdots C_i^{f_{i,1}(\sigma)} = 1.$$

Then we have

$$\left(C_1^{f_{1,1}(\sigma)} \cdots C_i^{f_{i,1}(\sigma)}\right)^{1-\tau} = 1$$

i. e.

$$C_1^{f_{1,1}(\sigma)-f_{1,1}(\sigma^a)} \cdots C_i^{f_{i,1}(\sigma)-f_{i,1}(\sigma^a)} = 1.$$

Applying here lemma 4, we obtain

$$\left(C_2^{f_{2,2}(\sigma)} \cdots C_i^{f_{i,2}(\sigma)}\right)^{\sigma-\sigma^a} = 1$$

$$\left(C_2^{f_{2,2}(\sigma)} \cdots C_i^{f_{i,2}(\sigma)}\right)^{1-\sigma^{a-1}} = 1.$$

Because of lemma 2 and the assumption of the theorem this reduces to

$$C_2^{f_{2,2}(\sigma)} \cdots C_i^{f_{i,2}(\sigma)} = 1.$$

By repeating this procedure $(i - 1)$-times, we obtain

$$C_i^{f_{i,i}(\sigma)} = 1$$

that is

$$C_i = 1. \quad \text{q. e. d.}$$

---

[4]) In particular it follows from (6) that $f_{n,i}(1)$ is equal to the number of positive integral solutions of the Diophantine equation

$$X_1 + X_2 + \cdots + X_i = n.$$

§ 3. The main theorem.

We shall now deal with the arithmetical structure of the field L/P. First suppose that K/P is unramified. Because of theorem 2 the assumption $a_{K/P} = 1$ means that the class number of P is l and K is the absolute class field of P. In this case L is the second class field of P, and L/Ω is clearly unramified. Next consider the case where K/P is ramified. In this case, it is necessary for $a_{K/P} = 1$ that $h_P = 1$ (theorem 2). Because of lemma 2 the group $\mathfrak{G}$ is constituted by the subgroup $\mathfrak{S}$, the subset $\{\tau, \ldots, \tau^{l-1}\}$ and its conjugates. Hence every subgroup of $\mathfrak{G}$ of order l is a conjugate of $\mathfrak{T}$, and every intermediate field Ω' of L/P such that $(\mathsf{L} : \Omega') = l$ is conjugate to Ω. Now by our assumption, there exists a prime divisor of L with the ramification exponent l. The inertia field of such a prime divisor must be conjugate to Ω (over P) and so L/Ω must be ramified.

Denote by $\mathfrak{K}_{\mathsf{k}}$ the absolute class group of a field k, and by $\mathfrak{K}_{\mathsf{k}'/\mathsf{k}}$ the subgroup of $\mathfrak{K}_{\mathsf{k}'}$ of all the ambigous classes with respect to a cyclic extension k'/k. Moreover denote by M and Φ_i the absolute class field of L and of Ω^{σ^i} respectively. With these notations our main theorem is stated as follows:

Main theorem. 1. *The fields* $\mathsf{L}\Phi_0, \mathsf{L}\Phi_1, \ldots, \mathsf{L}\Phi_{l-1}$ *are independent over* L, *and their composite* $\mathsf{L}\Phi_0\Phi_1 \cdots \Phi_{l-1}$ *coincides with* M.

2. $\mathfrak{K}_{\mathsf{L}} = \mathfrak{K}_{\mathsf{L}/\Omega} \times \mathfrak{K}_{\mathsf{L}/\Omega^\sigma} \times \cdots \times \mathfrak{K}_{\mathsf{L}/\Omega^{\sigma^{l-1}}}$ *(direct product), and* $\mathfrak{K}_{\mathsf{L}/\Omega^{\sigma^i}}$ *is isomorphic to a subgroup of* $\mathfrak{K}_\Omega$ *of index* h_P.

3. $h_{\mathsf{L}} = \left(\dfrac{h_\Omega}{h_P}\right)^l$.

Here $h_P = 1$, *if* K/P *is ramified, and* $h_P = l$, *if* K/P *is unramified.*

Proof. We shall apply the results of § 2 to the group $\mathfrak{K}_{\mathsf{L}}$ with the operator domain $\mathfrak{G}$. Because of theorem 1, $(\mathfrak{K}_{\mathsf{L}})_\sigma = 1$, i. e. thea ssumption of theorem 3 and theorem 4 is satisfied for $\mathfrak{K}_{\mathsf{L}}$. First we shall prove that the field $M' = \mathsf{L}\Phi_0\Phi_1 \cdots \Phi_{l-1}$ coincides with M. As M' is an unramified Abelian extension of L, there corresponds to M' in the sense of the class field theory a subgroup of $\mathfrak{K}_{\mathsf{L}}$. By the theorem of translation in the class field theory this is nothing else than $\bigcup_{i=0}^{l-1} \sigma^{-i}\tau\sigma^i(\mathfrak{K}_{\mathsf{L}})$. Because of theorem 3 this reduces to the indentity of $\mathfrak{K}_{\mathsf{L}}$, i. e. the principal class of L. Hence M' coincides with the absolute class field M of L. In particular we have

$$h_{\mathsf{L}} \leqq \left(\frac{h_\Omega}{h_P}\right)^l \tag{8}$$

since $(\mathsf{L}\Phi_i : \mathsf{L}) = \dfrac{h_\Omega}{h_P}$.

On the other hand, it follows directly from theorem 4 that the product of subgroups $\mathfrak{K}_{\mathsf{L}/\Omega}, \mathfrak{K}_{\mathsf{L}/\Omega^\sigma}, \ldots, \mathfrak{K}_{\mathsf{L}/\Omega^{\sigma^{l-1}}}$ of $\mathfrak{K}_{\mathsf{L}}$ is direct. From this and from theorem 2 we obtain

$$h_{\mathsf{L}} \geqq \left(\frac{h_\Omega}{h_P}\right)^l \tag{9}$$

taking account of the remark at the beginning of this section. Thus equality must hold in (8) and in (9). Then all the assertions of the theorem are direct consequences of what has been shown. q. e. d.

Corollary. *Suppose that the absolute class field* K *of an algebraic number field* P *is cyclic and of prime degree* l *over* P *and that the absolute class field* L *of* K *is also cyclic over* K. *If the absolute class field* M *of* L *does not coincide with* L, *it is no longer cyclic over* L. *The Galois group of* M/L *is the direct product of* l *mutually isomorphic Abelian groups.*

Now let us consider the case where K/P is ramified. $h_P = 1$ was a necessary condition for $a_{K/P} = 1$. Put $o = 1$ if P contains a primitive l-th root of unity, and $o = 0$ otherwise. As ε^l is the norm of ε with respect to K/P for any unit ε in P, we have

$$(\varepsilon : N(\Theta)) \leqq l^{r+o}$$

where r is the rank of the unit group of P. Thus we obtain another necessary condition for $a_{K/P} = 1$, that is

$$e \leqq r + 1 + o. \tag{10}$$

In particular, if $h_P = 1$ and $e = 1$, we have $a_{K/P} = 1$ directly from (1). Now we consider the case where only one infinite prime divisor ramifies in K. (In this case l must be $= 2$.) Then, as -1 cannot be the norm of any element in K, we have $(\varepsilon : N(\Theta)) \geqq 2$. Hence from (1) and from theorem 2 it follows that at least one finite prime divisor must ramify in K, and that, if the number of such divisors is 1, we have $a_{K/P} = h_P$. Therefore it is sufficient for $a_{K/P} = 1$ that $h_P = 1$ and that only one finite and at most one infinite prime divisors ramify in K.

§ 4. The structure of the unit group of the absolute class field.

As is well-known, we can deduce from the theory of Artin's L-series and from the residue formula of Dedekind's zeta-functions a relation between the class numbers and the regulators of intermediate fields of L/P. Applying the main theorem to this relation, we get some relations between the regulators of intermediate fields. This gives an interesting assertion on the structure of the unit group of L in the case where P is the rational number field Q and K is an imaginary quadratic field (theorem 5).

For any field k, denote by R_{k} and w_{k} the regulator and the number of the roots of unity in it respectively. Applying the formula (12) in Kuroda [5] to the character relation of lemma 3, we have

$$\frac{h_{\mathsf{L}} R_{\mathsf{L}}}{w_{\mathsf{L}}} = \frac{h_{\mathsf{K}} R_{\mathsf{K}}}{w_{\mathsf{K}}} \left(\frac{h_\Omega R_\Omega w_P}{h_P R_P w_\Omega} \right)^l. \tag{11}$$

In the case where $P = \mathsf{Q}$, K does not contain any root of unity except ± 1 [5]), so does L neither. In this case, the quantity $\mathbf{J}$ defined in Brauer [6] is very easily obtained from the main theorem;

$$\mathbf{J} = \frac{h_{\mathsf{L}}}{h_K h_\Omega^l} \cdot \frac{l \cdot q^l}{lq} = \frac{q^{l-1}}{h_K} = q^{l-2}. \tag{12}$$

Now under the assumption that $P = \mathsf{Q}$, (11) is simplified by the main theorem to

$$R_{\mathsf{L}} = q \cdot R_K R_\Omega^l. \tag{13}$$

In particular, for an imaginary quadratic field K we obtain

$$R_{\mathsf{L}} = q \cdot R_\Omega^2. \tag{14}$$

From (14) follows

Theorem 5. *Let K be an imaginary quadratic field with a prime discriminant, and assume that the absolute class field L of K is cyclic over K. Then, if $\{\varepsilon_1, \ldots, \varepsilon_m\}$ is a system of fundamental units of the maximally real subfield of L, $\{\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_1^\sigma, \ldots, \varepsilon_m^\sigma\}$ forms a system of fundamental units of L. Here σ is a generating substitution of L/K.*

[5]) We exclude the trivial cases of $\mathsf{Q}(\sqrt{-1})$ and $\mathsf{Q}(\sqrt{-3})$.

Proof. We can take as τ the involution of complex numbers mapping any complex number to its complex conjugate. Then Ω is the maximally real subfield of L, and all of its conjugates are imaginary. Hence we have

$$m = \frac{q-1}{2}.$$

Now we compute the regulator $R[\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_1^\sigma, \ldots, \varepsilon_m^\sigma] = R[\varepsilon_i, \varepsilon_i^\sigma]$ of the unit system $\{\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_1^\sigma, \ldots, \varepsilon_m^\sigma\}$ in L

$$R[\varepsilon_i, \varepsilon_i^\sigma] = \left|\begin{array}{c|c} \cdots 2\log|\varepsilon_i^\sigma|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^2}|\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^m}|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^{m+1}}|\cdots \\ \hline \cdots 2\log|\varepsilon_i^{\sigma^{m+1}}|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^{m+2}}|\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^{2m}}|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^{2m+1}}|\cdots \end{array}\right|.$$

As $|\varepsilon_i^{\sigma^j}| = \overline{|\varepsilon_i^{\sigma^{q-j}}|} = |\varepsilon_i^{\sigma^{q-j}}|$, we have

$$R[\varepsilon_i, \varepsilon_i^\sigma] = \left|\begin{array}{c|c} \cdots 2\log|\varepsilon_i^\sigma|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^2}|\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^m}|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^{m+1}}|\cdots \\ \hline \cdots 2\log|\varepsilon_i^{\sigma^m}|\cdots & \cdots 2\log|\varepsilon_i^{\sigma^{m-1}}|\cdots \\ \cdots\cdots\cdots & \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma}|\cdots & \cdots 2\log|\varepsilon_i|\cdots \end{array}\right|$$

$$= \left|\begin{array}{c|c} \cdots 2\log|\varepsilon_i^\sigma|\cdots & \\ \cdots\cdots\cdots & * \\ \cdots 2\log|\varepsilon_i^{\sigma^m}|\cdots & \\ \hline 0 & \cdots 2\log|\varepsilon_i^{\sigma^{m-1}-\sigma^{m+1}}|\cdots \\ & \cdots\cdots\cdots \\ & \cdots 2\log|\varepsilon_i^{1-\sigma^2}|\cdots \end{array}\right|$$

$$= \pm R_D \times \left|\begin{array}{c} \cdots 2\log|\varepsilon_i^{\sigma^{m-1}-\sigma^m}|\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^{m-2}-\sigma^m}|\cdots \\ \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{1-\sigma^2}|\cdots \end{array}\right|.$$

Put R = the determinant of the second factor on the right hand side. Subtracting the first row of R from the 2nd, the 2nd from the 3rd, . . . , the $(m-1)$-th from the m-th, we get

$$R = \left|\begin{array}{c} \cdots 2\log|\varepsilon_i^{\sigma^{m-1}-\sigma^m}|\cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^{m-2}-\sigma^{m-1}}|\cdots \\ \cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{1-\sigma}|\cdots \end{array}\right|.$$

Then adding to the first row j-times of the j-th row ($j = 2, \ldots, m$) and noticing that $|\varepsilon_i|\,|\varepsilon_i^{\sigma+\cdots+\sigma^m}|^2 = 1$, we obtain

$$R = \begin{vmatrix} \cdots 2\log|\varepsilon_i^{m-(\sigma+\cdots+\sigma^m)}| \cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^{m-2}-\sigma^{m-1}}| \cdots \\ \cdots\cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{1-\sigma}| \cdots \end{vmatrix} = q \begin{vmatrix} \cdots \log|\varepsilon_i| \cdots \\ \cdots 2\log|\varepsilon_i^{\sigma^{m-2}-\sigma^{m-1}}| \cdots \\ \cdots\cdots\cdots\cdots \\ \cdots 2\log|\varepsilon_i^{1-\sigma}| \cdots \end{vmatrix} = \pm q R_\Omega.$$

Thus we get from this and from (14)

$$\begin{aligned} R[\varepsilon_i, \varepsilon_i^\sigma] &= \pm R_\Omega \cdot R = \pm q \cdot R_\Omega^2 \\ &= \pm R_{\mathsf{L}}, \end{aligned}$$

Which means that $\{\varepsilon_1, \ldots, \varepsilon_m, \varepsilon_1^\sigma, \ldots, \varepsilon_m^\sigma\}$ is in fact a system of fundamental units in L. q. e. d.

In the case where $h_K = q = 3$, theorem 5 means that we can take a system of Minkowski units as a system of fundamental units in L [6]).

§ 5. Examples.

As an application of our result we shall give some numerical examples of non-Abelian Galois fields of class number 1. Throughout this section we shall take as the ground field the rational number field Q. Let L_1 be the splitting field of the equation

$$x^3 - x + 1 = 0. \tag{15}$$

As the discriminant of (15) is -23, (15) has one real and two imaginary roots, the Galois group of L_1 is isomorphic to the symmetric group of degree 3, and the imaginary quadratic field contained in L_1 is $K_1 = \mathsf{Q}(\sqrt{-23})$. Because the class number of $\mathsf{K}_1 = 3$ and it is easily verified that L_1/K_1 is unramified, L_1 is just the absolute class field of K_1. Let Ω_1 be the real cubic subfield of L_1. Every absolute class of Ω_1 contains an integral ideal whose norm does not exceed $\sqrt{23} = 4, \ldots$. As

$$n^3 - n + 1 = (n-1)\,n(n+1) + 1 \equiv 1 \pmod 6$$

for any rational integer n, both of (2) and (3) are prime ideals in Ω_1 and their norms exceed 5. Hence Ω_1 has the class number 1. Therefore the class number of L_1 is also equal to 1 because of the main theorem.

In the same way we can prove that the splitting fields of the equations

$$x^3 + x + 1 = 0 \tag{15'}$$

$$x^3 + 2x + 1 = 0 \tag{15''}$$

have the class number 1. They are the absolute class fields of the quadratic fields $K_2 = \mathsf{Q}(\sqrt{-31})$ and $K_3 = \mathsf{Q}(\sqrt{-59})$ of class number 3 respectively. For example we shall prove that the real cubic subfield Ω_2 of the splitting field of (15′) has the class number 1. Every absolute class of Ω_2 contains an integral ideal whose norm does not exceed $\sqrt{31} = 5, \ldots$. As $x^3 + x + 1$ is irreducible modulo 2 and modulo 5, both of (2) and (5) are prime ideals in Ω_2, and their norms exceed 6. Let Θ be the real root of (15′). We have

$$\begin{aligned} 3 &= 2 - \Theta - \Theta^3 \\ &= (1-\Theta)\,(2 + \Theta + \Theta^2). \end{aligned}$$

[6]) By Siegel's theorem, K with $h_K = q = 3$ i. e. imaginary quadratic fields with the class number 3 exist only in finite number.

Neither $1 - \Theta$ nor $2 + \Theta + \Theta^2$ is a unit in Ω_2, because none of their norms is equal to ± 1. Hence the two prime divisors of 3 in Ω_2 are principal ideals. Thus Ω_2 in fact must have the class number 1.

Finally we shall give an example of a quadratic field whose absolute class field has the degree 3 and the class number larger than 1. Let M_4 be the splitting field of the equation

$$y^4 + y - 1 = 0. \tag{16}$$

The discriminant of (16) is -283. Without difficulty we can verify that the Galois group of M_4 is isomorphic to the symmetric group of degree 4, and M_4 contains the splitting field L_4 of the cubic equation

$$x^3 + 4x + 1 = 0. \tag{17}$$

The Galois group of L_4 is isomorphic to the symmetric group of degree 3 and L_4 is an unramified extension of the imaginary quadratic field $K_4 = \mathsf{Q}(\sqrt{-283})$. As the class number of $K_4 = 3$, L_4 is the absolute class field of K_4. We shall prove that $\mathsf{M}_4/\mathsf{L}_4$ is also unramified[7]), which implies that the class number of $\mathsf{L}_4 \geqq 4$. It suffices to prove that the prime 283 ramifies in M_4 with the ramification exponent 2. Let N_4 be the subfield of M_4 generated by one root of (16). As the discriminant of N_4 is clearly -283, prime ideal decomposition of 283 in N_4 must take form $(283) = \mathfrak{p}^3\mathfrak{q}$ or $(283) = \mathfrak{p}^2\mathfrak{q}\mathfrak{q}'$. Anyhow a prime divisor of $\mathfrak{q}$ in M_4 has the ramification exponent 2 with reference to N_4, therefore with reference to Q. This completes our proof.

References.

[1] *H. Hasse*, Bericht über neuere Untersuchungen und Probleme aus der Theorie der algebraischen Zahlkörper. I (1926), Ia (1927), II (1930), Jahresberichte der D. M. V.

[2] *M. Moriya*, Über die Klassenzahl eines relativzyklischen Zahlkörpers von Primzahlgrad, Proc. Jap. Acad. **6** (1930).

[3] *M. Moriya*, Über die Klassenzahl eines relativzyklischen Zahlkörpers von Primzahlgrad, Jap. Journ. Math. **10** (1933).

[4] *H. Nehrkorn*, Über absolute Idealklassengruppe und Einheiten in algebraischen Zahlkörpern, Abh. Math. Sem. Univ. Hamburg **9** (1933).

[5] *S. Kuroda*, Über die Klassenzahl algebraischer Zahlkörper, Nagoya Math. J. **1** (1950).

[6] *R. Brauer*, Beziehungen zwischen Klassenzahlen von Teilkörpern eines galoisschen Körpers, Math. Nachr. **4** (1951).

[7] *T. Tannaka*, On the generalized principal ideal theorem, Proc. Int. Symp. Alg. Nb. Th. Tokyo-Nikko. (1955).

[8] *M. Ishida*, On the divisibility of Dedekind's zeta-functions, Proc. Jap. Acad. **33** (1957).

[7]) In truth, M_4 is just the absolute class field of L_4.

Eingegangen 28. April 1959.

Contribution to the theory of maximal, normal operators in a separable and complete Hilbert-Hermite-space. II.

An item of operational calculus and its application to the resolvent and spectrum.

By *Otton Martin Nikodým*, Kenyon College, Gambier (Ohio, U.S.A.).

We take over the notions defined in our paper (1), so we ask the reader to be familiar with it. The present is its continuation, or rather a supplement to the theory of the operator $\varphi(\mathfrak{N})$ where $\varphi(z)$ is a $\mathfrak{N}$-measurable complex valuable function of the complex variable z, defined on an $\mathfrak{N}$-measurable subset of the complex plane P, and where $\mathfrak{N}$ is a maximal normal operator in a Hilbert-Hermite-space $\mathfrak{H}$. Our purpose is to study the most general operators $\Psi(\varphi(\mathfrak{N}))$, $\varphi(\Psi(\mathfrak{N}))$ and $(\varphi\Psi)(\mathfrak{N})$ — which are not identical and need quite subtle reasonings — and apply the results to the general theory of the eigenvalue problem, of the resolvent and of the spectrum of $\mathfrak{N}$. We shall show the great adaptability of our theory (1) to that topic, and we shall get a classification[1]) of points of the spectrum of normal maximal operators, which seems to be more subtle than the usual one.

We shall be quite explicit in arguments, because the topic, spoken of, is just barely touched in (1). We shall take care of our argument to be valid even in the case when $\mathfrak{H}$ is confined to the single vector $\vec{O}$, and also in the case where the operator $\mathfrak{N}$ is trivial, (denoted by $\mathfrak{O}$), i. e. carrying every vector of the space $\mathfrak{H}$ into the null-vector. Indeed the above is needed in many circumstances. Our notations differ slightly from those used in (1); they will be, however, explained in the sequal. If we like to emphasize that a letter, say p, is a variable we shall provide it with a dot: $\dot{p}$.

1. Let $\mathfrak{H}$ be a separable and complete Hilbert-Hermite-space [H. H.-space] which may be finite or infinite dimensional or even be confined to the single vector $\vec{O}$. Let $\mathfrak{N}$ be a maximal normal operator in $\mathfrak{H}$ which may be even $\mathfrak{O}$. The spectral tribe of $\mathfrak{N}$ will be denoted by $\mathfrak{T}_N$, its borelian spectral tribe by $\mathfrak{T}_N^B$ [2]). The tribe $\mathfrak{T}_N^B$ may be saturated or not.

Whatever the case may be, there exists an infinite sequence $\{\mathfrak{H}_n\}$ of mutually disjoint and mutually compatible spaces, which are also compatible with all spaces of $\mathfrak{T}_N^B$, and such that 1) the tribe obtained from $\mathfrak{T}_N^B$ by the adjunction of $\mathfrak{H}_n$ be saturated, 2) $\sum_{n=1}^{\infty} \mathfrak{H}_n = \mathfrak{H}$ [3]). The sequence $\mathfrak{H}_n$ will be termed *saturating sequence.*

If we consider the most general case, some of $\mathfrak{H}_n$ may be confined to the single vector $\vec{O}$. The saturated tribe will be denoted by $\overline{\mathfrak{T}}_N^B$. Let $\vec{\omega}$ be a generating vector of $\mathfrak{H}$

---

[1]) It follows the pattern of classification, given in (4).

[2]) See (1), p. 131; denoted by (s).

[3]) See (1), p. 132. 5, and p. 137. 12.

with respect to $\overline{\mathfrak{T}}_N^B$, and such that $\vec{\omega} \in \mathfrak{D}\,\mathfrak{N}$, (domain of $\mathfrak{N}$). The measure $\mu(a)$ defined for all spaces $a \in \overline{\mathfrak{T}}_N^B$, by

$$(1) \qquad \mu(a) \underset{\mathrm{df}}{=} \| \operatorname{Proj}_a \vec{\omega} \|^2$$

is denumerably additive, finite, non negative and effective [4]).

The saturating sequence and the generating vector determine a canonical mapping $\mathfrak{C}$ of $\mathfrak{P}$ onto the H.H.-space $\mathcal{H}$ whose vectors are complex valued μ-square summable functions $X(\dot{p})$, defined almost μ-everywhere (a. e.) on the disjoint union V of an infinite denumerable number of exact copies V_n of the complex plane P. The measure μ on V is induced by (1) in the way explained in (1) [5]), and the μ-measurable subsets of V and the μ-measurable functions are taken modulo the ideal $\mathfrak{J}_\mu$ of subsets of V with μ-measure equal 0. We shall use the sign $=^\mu$ to denote the corresponding equality of sets and functions. The canonical mapping is isometric, homeomorphic and isomorphic. It carries the operator $\mathfrak{N}(\vec{X})$ into the operator [6])

$$(2) \qquad \mathfrak{N}^*(X(\dot{p})) =^\mu (\operatorname{Num} \dot{p}) \cdot X(\dot{p}),$$

where $\mathfrak{C}$ carries $\vec{X}$ into $X(\dot{p})$ and where the domain of (2) is *maximal*, which means that $\mathfrak{D}\,\mathfrak{N}$ is composed of all functions $X(\dot{p}) \in \mathcal{H}$ such that $(\operatorname{Num} \dot{p})\, X(\dot{p}) \in \mathcal{H}$.

2. Let $a \in \overline{\mathfrak{T}}_N^B$. The $\mathfrak{C}$-image of a vector $\vec{X} \in a$ is a function $X(\dot{p})$ which vanishes μ — a. e. on a μ-measurable subset b of V, the same for all vectors $\vec{X} \in a$. The complement $a \underset{\mathrm{df}}{=} V - b$ is termed $\hat{\mathfrak{C}}$-*image of* a. The correspondence $\hat{\mathfrak{C}}$ is generated by $\mathfrak{C}$, and conversely. $\hat{\mathfrak{C}}$ will be termed the *correspondence associate to* $\mathfrak{C}$. It transforms spaces belonging to the saturated tribe $\overline{\mathfrak{T}}_N^B$ into μ-measurable subsets of V, taken module $\mathfrak{J}_\mu$. The correspondence $\hat{\mathfrak{C}}$ is an isometric isomorphism preserving finite and denumerable somatic operations [7]).

3. The measure μ on $\overline{\mathfrak{T}}_N^B$ induces a denumerably additive measure, on the complex plane P, which corresponds to the measure on $\mathfrak{T}_N^B$ [8]). The corresponding measurable sets constitute an assemblage which does not depend on the choice of the canonical correspondence, but only on $\mathfrak{N}$. We call the sets $\mathfrak{N}$-measurable. $\mathfrak{J}_N$ will denote the ideal of all $\mathfrak{N}$-measurable subsets of P with measure 0. The corresponding equality of sets will be denoted $=^N$.

In what follows the μ-measurable subsets of V will always be taken module $\mathfrak{J}_\mu$, and the $\mathfrak{N}$-measurable subsets of P will be taken modulo $\mathfrak{J}_N$, and the same will be for corresponding functions $X(\dot{p})$ and $\varphi(\dot{z})$.

4. If $\varphi(\dot{z})$ is an $\mathfrak{N}$-measurable, complex valued function defined $\mathfrak{N}$-a. e. on P, then the operator $\varphi(\mathfrak{N})$ is defined as the $\mathfrak{C}^{-1}$-image of the operator

$$[\varphi(\mathfrak{N})]\, X(\dot{p}) =^\mu \cdot\, \varphi(\operatorname{Num} \dot{p}) \cdot X(\dot{p})$$

[4]) Even if the tribe is trivial i. e. its unit and zero coincide, the measure is effective, because $\mu(a) = 0$ implies $a = 0$.

[5]) See (1), p. 132. 5, p. 131. 4. The n-th copy V_n of P is the set of all ordered couples (z, n) where $n = 1, 2, \ldots$ and z is a complex number. We define $\operatorname{Num}(z, n) \underset{\mathrm{df}}{=} z$. If $E \subseteq P$, then by Point E we understand the set $\bigcup_{n=1}^{\infty} \{(z, n) \mid z \in E\}$. It is a subset of $V = \bigcup_{n=1}^{\infty} V_n$.

[6]) See (1), 5.

[7]) See (1), we used the sign C for both $\mathfrak{C}$ and $\hat{\mathfrak{C}}$.

[8]) See (1), p. 136. 9.

taken with maximal domain. $\varphi(\mathfrak{N})$ does not depend on the choice of the canonical mapping $\mathfrak{C}$ [9]).

5. It will be useful to denote by $\mathcal{H}^V(a)$ the H. H.-space composed of all μ-square summable functions $X(\dot{p})$ defined μ-a. e. in V, but vanishing μ-a. e. on $\mathrm{co}_V\, a = V - a$. Here a is supposed to be a μ-measurable subset of V. Its $\mathfrak{C}^{-1}$-image, $\boldsymbol{a} = \hat{\mathfrak{C}}^{-1} a$ is a space of $\overline{\mathfrak{T}}_N^B$.

6. We shall define $\varphi(\mathfrak{N})$ in the case where $\varphi(z)$ is defined only on a $\mathfrak{N}$-measurable subset E of P [10]).

Denote by $\varphi^P(\dot{z})$ the function defined on P by

$$\varphi^P(z) = \begin{cases} \varphi(z) & \text{for } z \in E, \\ 0 & \text{for } z \notin E. \end{cases}$$

We can prove that the operator

$$\boldsymbol{e} \restriction \varphi^P(\mathfrak{N})\ ^{11}),$$

where $\boldsymbol{e} \underset{\mathrm{df}}{=} \hat{\mathfrak{C}}^{-1}$ (Point E), does not depend on the choice of $\{\mathfrak{P}_n\}$, neither on $\vec{\omega}$, i. e. it does not depend on $\mathfrak{C}$, but only on $\mathfrak{N}$ and φ.

We define $\varphi(\mathfrak{N}) \underset{\mathrm{df}}{=} \boldsymbol{e} \restriction \varphi^P(\mathfrak{N})$. We can prove that

$$\varphi(\mathfrak{N}) \equiv \varphi(\boldsymbol{e} \restriction \mathfrak{N}).$$

The above takes place even if $\mathfrak{P}$ is confined to the single vector $\vec{O}$ and also if $E =^N \emptyset$. The operator $\varphi(\mathfrak{N})$ operates in $\boldsymbol{e}$ only, i. e. it carries some vectors of $\boldsymbol{e}$ into vectors of $\boldsymbol{e}$. Hence the $\mathfrak{C}$-image of $\varphi(\mathfrak{N})$ operates on functions $X(\dot{p})$ belonging to $\mathcal{H}^V$ (Point E).

Remark. It does not operate on functions defined only on Point E. Consequently the $\mathfrak{C}$-image of $\varphi(\mathfrak{N})$ is the operator

$$(1) \qquad [\varphi(\mathfrak{N})]^* X(\dot{p}) =^\mu \varphi^P(\mathrm{Num}\, \dot{p}) \cdot X(\dot{p})$$

with maximal domain in $\mathcal{H}^V$ (Point E). Thus the domain is composed of all functions of $\mathcal{H}^V$ (Point E) such that

$$\varphi^P(\mathrm{Num}\, \dot{p})\ X(\dot{p}) \in \mathcal{H}^V(\mathrm{Point}\ E).$$

One can also say that (1) is restricted to $\mathcal{H}^V$ (Point E).

We shall study these operators.

7. Theorem. *If* 1) *E is an $\mathfrak{N}$-measurable subset of the plane P,*

2) *$\varphi(z)$ is defined in E only,*

3) *$\varphi(z) = 0$ on E,*

then $\varphi(\mathfrak{N}) = \boldsymbol{e} \restriction \mathfrak{D}$, *where* $\boldsymbol{e} = \mathfrak{C}^{-1}\mathcal{H}^V$ (Point E).

This operator carries every vector of $\boldsymbol{e}$ into $\vec{O}$.

8. Theorem. *If* 1. *E_1, E_2 are $\mathfrak{N}$-measurable subsets of P,*

2. $E_1 \cap E_2 \cdot =^N \cdot \emptyset$, $E \underset{\mathrm{df}}{=} E_1 \cup E_2$,

3. *$\varphi(z)$ is $\mathfrak{N}$-measurable and defined only on $E_1 \cup E_2 = E$,*

4. $\varphi_1(z) \underset{\mathrm{df}}{=} E_1 \restriction \varphi(z)$, $\varphi_2(z) \underset{\mathrm{df}}{=} E_2 \restriction \varphi(z)$,

5. $\boldsymbol{e}_1 \underset{\mathrm{df}}{=} \mathfrak{C}^{-1}\mathcal{H}^V$ (Point E_1), $\boldsymbol{e}_2 \underset{\mathrm{df}}{=} \mathfrak{C}^{-1}\mathcal{H}^V$ (Point E_2),

---

[9]) See (1), 7, 8.

[10]) Only mentioned in (1). p. 135, 9.

[11]) $\boldsymbol{e} \restriction$ means "restricted to the space $\boldsymbol{e}$".

then

1) $\mathsf{D}\varphi(\mathfrak{N}) = \mathsf{D}\varphi_1(\mathfrak{N}) \dot{+} \mathsf{D}\varphi_2(\mathfrak{N})$, *which means:*

$$\mathsf{D}\varphi(\mathfrak{N}) = \{\vec{X}_1 + \vec{X}_2 \mid \vec{X}_1 \in \mathsf{D}\varphi_1(\mathfrak{N}),\ \vec{X}_2 \in \mathsf{D}\varphi_2(\mathfrak{N})\},$$

2) $\varphi(\mathfrak{N}) \equiv \varphi_1(\mathfrak{N}) \dot{+} \varphi_2(\mathfrak{N})$, *which means:*

for every $\vec{X} \in \mathsf{D}\varphi(\mathfrak{N})$ *we have* $\varphi(\mathfrak{N})\,\vec{X} = \varphi_1(\mathfrak{N})\,\vec{X}_1 + \varphi_2(\mathfrak{N})\,\vec{X}_2$,

where $\vec{X}_1 \underset{\mathrm{df}}{=} \mathrm{Proj}_{e_1}\vec{X}$, $\vec{X}_2 \underset{\mathrm{df}}{=} \mathrm{Proj}_{e_2}\vec{X}$.

This is the "smallest" linear operator containing $\varphi_1(\mathfrak{N})$ and $\varphi_2(\mathfrak{N})$. We omit the proof, which is rather straightforward and does not need special artifices.

9. Theorem. *If* 1. E, F *are* $\mathfrak{N}$*-measurable subsets of* P,

2. $E \leq F$,

3. $\varphi(z)$ *is defined only on* E,

4. $\Psi(z)$ *is defined only on* F *and is* $\mathfrak{N}$*-measurable,*

4. $\varphi(z) = 0$ *on* E,

then

$$e \restriction \Psi(\mathfrak{N}) = \Psi(\mathfrak{N}) + \varphi(\mathfrak{N})\ ^{12)},$$

where the addition is the addition of operators.

10. Remark. If $e_1, e_2 \in \overline{\mathfrak{T}}_N^B$, then

$$e_1 \restriction \mathfrak{D} \dot{+} e_2 \restriction \mathfrak{D} \equiv (e_1 + e_2) \restriction \mathfrak{D}.$$

It is the smallest linear operator containing $e_1 \restriction \mathfrak{D}$ and $e_2 \restriction \mathfrak{D}$.

But we have

$$e_1 \restriction \mathfrak{D} + e_2 \restriction \mathfrak{D} \equiv (e_1 e_2) \restriction \mathfrak{D}.$$

11. Theorem. *If* 1. E_1, E_2 *are* $\mathfrak{N}$*-measurable subsets of the plane* P,

2. $E_1 \subseteq E$,

3. $\varphi(\dot{z})$ *is* $\mathfrak{N}$*-measurable and defined only on* E,

4. $\varphi(z) = 0$ on $E - E_1$,

5. $e \underset{\mathrm{df}}{=} \mathfrak{C}^{-1}\mathcal{H}^V(\text{Point } E)$, $e_1 \underset{\mathrm{df}}{=} \mathfrak{C}^{-1}\mathcal{H}^V(\text{Point } E_1)$,

6. $\varphi_1(\dot{z}) \underset{\mathrm{df}}{=} E_1 \restriction \varphi(\dot{z})$,

then

1) *the domain of* $\varphi(\mathfrak{N})$ *is* $\mathsf{D}\varphi_1(\mathfrak{N}) \dot{+} (e - e_1)$.

2) $\varphi_1(\mathfrak{N}) \equiv e_1 \restriction \varphi(\mathfrak{N})$,

3) $\varphi(\mathfrak{N}) \equiv \varphi_1(\mathfrak{N}) \dot{+} (e - e_1) \restriction \mathfrak{D}$.

Proof. Since $E_1 \cap (E - E_1) = \emptyset$, we get, by [8],

$$\mathsf{D}\varphi_1(\mathfrak{N}) \dot{+} \mathsf{D}\,[(E - E_1) \restriction \varphi]\,\mathfrak{N} = \mathsf{D}\varphi(\mathfrak{N}).$$

Since $\varphi(z) = 0$ on $E - E_1$, we get, by [7],

$$\mathsf{D}\,[(E - E_1) \restriction \varphi]\,(\mathfrak{N}) = e - e_1.$$

Hence

$$\mathsf{D}\varphi(\mathfrak{N}) = \mathsf{D}\varphi_1(\mathfrak{N}) \dot{+} (e - e_1). \tag{1}$$

By [8] we also have

$$\varphi(\mathfrak{N}) = \varphi_1(\mathfrak{N}) \dot{+} [(E - E_1) \restriction \varphi]\,(\mathfrak{N}).$$

---

[12]) According to definition admitted in (2), the domain of $\Psi(\mathfrak{N}) + \varphi(\mathfrak{N})$ is $\mathsf{D}\,\Psi(\mathfrak{N}) \cap \mathsf{D}\,\varphi(\mathfrak{N})$.

Hence, by [7],

$$\varphi(\mathfrak{N}) = \varphi_1(\mathfrak{N}) \dot{+} (e - e_1) \restriction \mathfrak{O}. \tag{2}$$

To prove item 2) we notice that the $\mathfrak{C}$-image of $e_1 \restriction \varphi(\mathfrak{N})$ is the restriction to $\mathcal{H}^V$ (Point E_1) of the $\mathfrak{C}$-image of the operator $\varphi(\mathfrak{N})$. Hence, it is the restriction to $\mathcal{H}^V$ (Point E_1) of the operator

$$[\varphi(\mathfrak{N})]^* X(\dot{p}) \cdot =^{\mu} \cdot \varphi(\text{Num } p) \cdot X(\dot{p})$$

which is considered with maximal domain in $\mathcal{H}^V$ (Point E). Now this restriction is identical with the operator, with maximal domain, which carries functions of $\mathcal{H}^V$ (Point E_1) into $\varphi_1(\text{Num } \dot{p}) \cdot X(\dot{p})$. But the last operator is the $\mathfrak{C}$-image of $(E_1 \restriction \varphi)(\mathfrak{N}) = \varphi_1(\mathfrak{N})$. Consequently $e_1 \restriction \varphi(\mathfrak{N}) = \varphi_1(\mathfrak{N})$. Q. E. D.

12. To simplify the symbolism, denote in general by $D^{\cdot}$ the set Point D, where D is an $\mathfrak{N}$-measurable subset of the complex plane P. In addition to that we shall use the signs "+" and "." for union and intersection of sets. The equality for $\mathfrak{N}$-measurable subsets of P will be $=^N$, and the sets will be taken modulo $\mathfrak{J}_N$. The equality for μ-measurable subsets of V will be $=^{\mu}$, and the sets will be taken modulo $\mathfrak{J}_{\mu}$. We shall use the ordinary sign "=" of equality in both cases unless we would like to emphasize the character of the equality: then we shall write $=^{\mu}$ or $=^N$. Under these agreements we have for $\mathfrak{N}$-measurable sets $\emptyset^{\cdot} = \emptyset$ on V, $P^{\cdot} = V$, $(E + F)^{\cdot} = E^{\cdot} + F^{\cdot}$, $(E \cdot F)^{\cdot} = E^{\cdot} \cdot F^{\cdot}$, $(E - F)^{\cdot} = E^{\cdot} - F^{\cdot}$, $(\text{co}_P E)^{\cdot} = \text{co}_V E^{\cdot}$. Thus the sets D and $D^{\cdot}$ are operation-isomorphic.

13. Let A, B, A_1, B_1 be $\mathfrak{N}$-measurable subsets of P, where $A_1 \subseteq A$, $B_1 \subseteq B$. Let $\varphi(\dot{z})$, $\Psi(\dot{z})$ be $\mathfrak{N}$-measurable complex valued functions of the complex variable $\dot{z}$, defined only on A, B respectively. We suppose that

$$\varphi(z) \neq 0 \text{ for } z \in A_1, \qquad \Psi(z) \neq 0 \text{ for } z \in B_1,$$
$$\varphi(z) = 0 \text{ for } z \in A - A_1, \quad \Psi(z) = 0 \text{ for } z \in B - B_1.$$

We shall consider the operators

$$\mathfrak{P} \underset{\text{df}}{=} \varphi(\mathfrak{N}), \ \mathfrak{Q} \underset{\text{df}}{=} \Psi(\mathfrak{N}),$$

and shall study the operators

$$\mathfrak{R}\vec{X} \underset{\text{df}}{=} \mathfrak{Q}(\mathfrak{P}\vec{X}), \text{ and } \mathfrak{S}\vec{X} \underset{\text{df}}{=} \mathfrak{P}(\mathfrak{Q}\vec{X}).$$

13. 1. The $\mathfrak{C}$-image of the operator $\mathfrak{P}$ is

$$\mathfrak{P}^*(X(\dot{p})) = \varphi^P(\text{Num } \dot{p}) \cdot X(\dot{p}) \tag{1}$$

with maximal domain in $\mathcal{H}^V(A^{\cdot})$.

The $\mathfrak{C}$-image of the operator $\mathfrak{Q}$ is

$$\mathfrak{Q}^*(X(\dot{p})) = \Psi^P(\text{Num } \dot{p}) \cdot X(\dot{p}) \tag{2}$$

with maximal domain in $\mathcal{H}^V(B^{\cdot})$.

The $\mathfrak{C}$-image of $\mathfrak{R}$ is

$$\mathfrak{R}^*(X(\dot{p})) = \mathfrak{Q}^*[\mathfrak{P}^*(X(\dot{p}))].$$

Its domain is composed of all functions $X(\dot{p})$, for which $\mathfrak{P}^*(X(\dot{p})) \in \mathfrak{D}\mathfrak{Q}^*$. More explicitely $\mathfrak{D}\mathfrak{R}^*$ is composed of all functions $X(\dot{p})$ for which

$$\mathfrak{P}^*(X(\dot{p})) \in \mathcal{H}^V(B^{\cdot}), \tag{3}$$

$$\Psi^P(\text{Num } \dot{p}) \cdot \mathfrak{P}^*(X(\dot{p})) \in \mathcal{H}^V(B^{\cdot}). \tag{4}$$

Since $\mathfrak{P}^*(X(\dot{p}))$ should be meaningful, therefore we must have

$$X(\dot{p}) \in \mathcal{H}^V(A^{\cdot}), \ \varphi^P(\text{Num } \dot{p}) \cdot X(\dot{p}) \in \mathcal{H}^V(A^{\cdot}).$$

Hence $\square\mathfrak{R}^*$ is composed of all functions $X(\dot{p})$ for which the following conditions are satisfied:

$$(5)\qquad \begin{cases} \text{(a)} & X(\dot{p}) \in \mathcal{H}, \\ \text{(b)} & \varphi^P(\text{Num } \dot{p}) \cdot X(\dot{p}) \in \mathcal{H}, \\ \text{(c)} & \Psi^P(\text{Num } \dot{p})\, \varphi^P(\text{Num } \dot{p})\, X(\dot{p}) \in \mathcal{H} \end{cases}$$

and

$$(6)\qquad \begin{cases} \text{(a)} & X(p) =^{\mu} 0 \text{ for } p \in \text{co } A^{\cdot}, \\ \text{(b)} & \varphi^P(\text{Num } p) \cdot X(p) =^{\mu} 0 \text{ for } p \in \text{co } A^{\cdot}, \\ \text{(c)} & \Psi^P(\text{Num } p)\, \varphi^P(\text{Num } p)\, X(p) =^{\mu} 0 \text{ for } p \in \text{co } B^{\cdot}, \\ \text{(d)} & \varphi^P(\text{Num } p)\, X(p) =^{\mu} 0 \text{ for } p \in \text{co } B^{\cdot}. \end{cases}$$

13. 2. We shall prove that under hypotheses (13) the conditions (6) are equivalent to the following one:

$$(7)\qquad X(p) =^{\mu} 0 \text{ for all } p \in [\text{co}\,(A \cdot B) - (A - A_1)]^{\cdot}.$$

Proof. Suppose that (7) holds true.

Since $\varphi^P(\text{Num } P) = 0$ in $A^{\cdot} - A_1^{\cdot}$, it follows that

$$(8)\qquad \varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } A^{\cdot} - A_1^{\cdot}.$$

Since $X(p) = 0$ in $\text{co}\,(A^{\cdot} \cdot B^{\cdot}) - (A^{\cdot} - A_1^{\cdot})$, we have

$$(9)\qquad \varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } \text{co}\,(A^{\cdot} \cdot B^{\cdot}) - (A^{\cdot} - A_1^{\cdot}).$$

From (8) and (9) we have:

$$\varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } \text{co } A^{\cdot} + \text{co } B^{\cdot} + (A^{\cdot} - A_1^{\cdot}).$$

Consequently

$$\varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } \text{co } A^{\cdot}, \text{ i. e. we have (6) (b)},$$

and

$$\varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } \text{co } B^{\cdot}, \text{ i. e. we have (6) (d)}.$$

Hence

$$\Psi^P(\text{Num } p) \cdot \varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ in } \text{co } B^{\cdot}, \text{ i. e. we have (6) (c)}.$$

From hypothesis (7) we have

$$X(p) = 0 \text{ for } p \in (\text{co } A^{\cdot} + \text{co } B^{\cdot})\ \text{co}\,(A^{\cdot}\ \text{co } A_1^{\cdot}) = (\text{co } A^{\cdot} + \text{co } B^{\cdot})\ (\text{co } A^{\cdot} + A_1^{\cdot}) \geqq \text{co } A^{\cdot}.$$

Hence $X(p) = 0$ for $p \in \text{co } A^{\cdot}$, i. e. we have (6) (a).

Thus we have proved that (7) implies (6).

13. 2. 1. Conversely, suppose that (6) holds true. We have $\varphi^P(z) = 0$ on $A - A_1$ and on $\text{co } A$. Hence $\varphi^P(\text{Num } p) = 0$ on $\text{co } A^{\cdot}$ and $A^{\cdot} - A_1^{\cdot}$, hence on

$$(10)\qquad \text{co } A^{\cdot} + (A^{\cdot} - A_1^{\cdot}) = \text{co } A_1^{\cdot}.$$

It follows that $\varphi^P(\text{Num } p) \cdot X(p) = 0$ on $\text{co } A_1^{\cdot}$. On the other hand we have, by (6) (d),

$$\varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ on } \text{co } B^{\cdot}.$$

Hence

$$\varphi^P(\text{Num } p) \cdot X(p) = 0 \text{ on } \text{co}\,(A_1^{\cdot} \cdot B^{\cdot}).$$

We have

$$(11)\qquad \text{co}\,(A_1^{\cdot} \cdot B^{\cdot}) = \text{co}\,(A_1^{\cdot} \cdot B^{\cdot}) \cdot A_1^{\cdot} + (\text{co } A_1^{\cdot} \cdot B^{\cdot})\ \text{co } A_1^{\cdot}.$$

The function $\varphi^P(\text{Num } p) \cdot X(p)$ vanishes on both terms of (11), but, since $\varphi^P(\text{Num } p) \neq 0$ on $A_1^{\cdot}$, therefore $X(p)$ must vanish on $\text{co}\,(A_1^{\cdot} \cdot B^{\cdot}) \cdot A_1^{\cdot}$. By (6) (a) we have

$$X(p) = 0 \text{ on } \text{co } A^{\cdot}.$$

Consequently

$$\text{(11a)} \quad X(p) = 0 \text{ on } \operatorname{co}(A_1^{\cdot} \cdot B^{\cdot}) \cdot A_1 + \operatorname{co} A^{\cdot}$$
$$= \operatorname{co} A_1^{\cdot} + (\operatorname{co} A_1^{\cdot} + \operatorname{co} B^{\cdot}) \cdot A_1 = \operatorname{co} A_1^{\cdot} + (A_1^{\cdot} - B^{\cdot}).$$

Now we can prove that

$$\text{(12)} \qquad \operatorname{co} A_1^{\cdot} + (A_1^{\cdot} - B^{\cdot}) = \operatorname{co}(A^{\cdot} \cdot B^{\cdot}) - (A^{\cdot} - A_1^{\cdot}),$$

just by taking the complements of both sides.

The complementary set is

$$\text{(13)} \qquad A^{\cdot} \cdot B^{\cdot} + (A^{\cdot} - A_1^{\cdot}) \text{ for both.}$$

Thus we have proved that (6) is equivalent to (7).

13. 2. 2. We have

$$A^{\cdot} B^{\cdot} + (A^{\cdot} - A_1^{\cdot}) = A_1^{\cdot} \cdot B^{\cdot} + (A^{\cdot} - A_1^{\cdot})$$

where both terms are disjoint. The domain of $\mathfrak{R}^*$ is contained in $\mathcal{H}^V[(A_1^{\cdot} \cdot B^{\cdot}) + (A^{\cdot} - A_1^{\cdot})]$. Let us consider the conditions (5) in [13. 1]. They can be replaced by similar conditions:

$$\text{(a}'\text{)} \qquad X(\dot{p}) \in \mathcal{H}^V[(A_1^{\cdot} B^{\cdot}) + (A^{\cdot} - A_1^{\cdot})],$$
$$\text{(b}'\text{)} \qquad X(p) \cdot \varphi^P(\operatorname{Num} \dot{p}) \in \mathcal{H}^V(A_1^{\cdot} B^{\cdot}) + \mathcal{H}^V(A^{\cdot} - A_1^{\cdot}),$$
$$\text{(c}'\text{)} \qquad X(p) \cdot \Psi^P(\operatorname{Num} \dot{p})\, \varphi^P(\operatorname{Num} \dot{p}) \in \mathcal{H}^V(A_1^{\cdot} B^{\cdot}) + \mathcal{H}^V(A^{\cdot} - A_1^{\cdot}).$$

Since

$$\varphi^P(z) = 0 \text{ on } A - A_1,$$

the condition (b′) is equivalent to

$$X(p)\, \varphi^P(\operatorname{Num} p) \in \mathcal{H}^V(A_1^{\cdot} B^{\cdot}).$$

Concerning (c′) we have

$$A_1 B = A_1 B_1 + A_1(B - B_1)$$

where the terms are disjoint. Hence

$$A_1 B + (A - A_1) = A_1 B_1 + A_1(B - B_1) + (A - A_1).$$

Since $\Psi^P = 0$ on $B - B_1$, and $\varphi^P = 0$ on $A - A_1$ the condition (c′) is equivalent to

$$X(p)\, \varphi^P(\operatorname{Num} p)\, \Psi^P(\operatorname{Num} p) \in \mathcal{H}^V(A_1^{\cdot} B_1^{\cdot}).$$

Thus the conditions (5) and (6) in [13. 1] are equivalent to the following ones:

$$\text{(14)} \qquad \begin{cases} X(\dot{p}) \in \mathcal{H}^V[A^{\cdot} B^{\cdot} + (A^{\cdot} - A_1^{\cdot})], \\ X(\dot{p})\, \varphi^P(\operatorname{Num} \dot{p}) \text{ is } \mu\text{-square summable on } A_1^{\cdot} B^{\cdot}, \\ X(\dot{p})\, \varphi^P(\operatorname{Num} p)\, \Psi^P(\operatorname{Num} p) \text{ is } \mu\text{-square summable on } A_1^{\cdot} B_1^{\cdot}. \end{cases}$$

Since the set $AB + (A - A_1)$ is the union of the following three disjoint parts: $A_1 B_1 + A_1(B - B_1) + (A - A_1)$, and we have $A_1 B = A_1 B_1 + A_1(B - B_1)$, we can say that the domain of $\mathfrak{R}^*$ is composed of all functions $X(\dot{p})$ belonging to $\mathcal{H}^V[A^{\cdot} B^{\cdot} + (A^{\cdot} - A_1^{\cdot})]$ for which $X(\dot{p}) \cdot \varphi^P(\operatorname{Num} \dot{p})$ is μ-square summable on $A_1^{\cdot} B_1^{\cdot}$ and on $A_1^{\cdot}(B^{\cdot} - B_1^{\cdot})$, and $X(\dot{p})\, \varphi^P(\operatorname{Num} \dot{p})\, \psi^P(\operatorname{Num} \dot{p})$ is μ-square summable on $A_1^{\cdot} B_1^{\cdot}$.

Thus $\mathfrak{R}^*$ can be decomposed into three parts:

1) The operator $\mathfrak{D}^*$ in $\mathcal{H}^V(A^{\cdot} - A_1^{\cdot})$;
2) the operator $\mathfrak{D}^*$ in $\mathcal{H}^V(A_1^{\cdot}(B^{\cdot} - B_1^{\cdot}))$ confined to functions belonging to $\mathfrak{C}[\varphi(\mathfrak{R})]^*$;
3) the operator carrying $X(\dot{p})$ into $\varphi^P(\operatorname{Num} \dot{p})\, \Psi^P(\operatorname{Num} \dot{p}) \cdot X(\dot{p})$ confined to $\mathcal{H}^V(A_1^{\cdot} B_1^{\cdot})$ and to functions of $\mathfrak{C}[\varphi(\mathfrak{R})]^*$.

The function $\xi(z) \underset{\mathrm{df}}{=} \varphi(z) \cdot \Psi(z)$ is defined in $A \cdot B$ and does not vanish in $A_1 \cdot B_1$. Let us denote the operator $\xi(\mathfrak{R})$ by $(\varphi\Psi)(\mathfrak{R})$. Its $\mathfrak{C}$-image is the operator $[\xi(\mathfrak{R})]^* X(\dot{p}) =^{\mu} \Psi^P(\mathrm{Num}\,\dot{p})\, \varphi^P(\mathrm{Num}\,\dot{p})\, X(\dot{p})$ with maximal domain $\mathcal{H}^V(A \cdot B \cdot)$. Having that, and if we take the $\mathfrak{C}^{-1}$-image, we get the following result, showing the decomposition of $\mathfrak{R}$ into three parts:

$$(15)\quad \begin{cases} \Psi(\varphi(\mathfrak{R})) = [a_1 b_1 \frown \mathsf{D}\,\varphi(\mathfrak{R})] \restriction (\Psi\varphi)(\mathfrak{R}) \\ \qquad\qquad \dotplus\, a_1(b - b_1) \frown \mathsf{D}\,\varphi(\mathfrak{R}) \restriction \mathfrak{D} \dotplus (a - a_1) \restriction \mathfrak{D}. \\ \mathsf{D}\,\Psi(\varphi(\mathfrak{R})) = [a_1 b_1 \frown \mathsf{D}\,\varphi(\mathfrak{R}) \frown \mathsf{D}\,(\Psi\varphi)(\mathfrak{R})] \\ \qquad\qquad + [a_1(b - b_1) \frown \mathsf{D}\,\varphi(\mathfrak{R})] + (a - a_1). \end{cases}$$

A similar result we have for $\mathfrak{S} = \varphi(\Psi(\mathfrak{R}))$:

$$(16\mathrm{a})\quad \begin{cases} \varphi(\Psi(\mathfrak{R})) = [a_1 b_1 \frown \mathsf{D}\,\Psi(\mathfrak{R})] \restriction (\varphi\Psi)(\mathfrak{R}) \\ \qquad\qquad \dotplus\, [b_1(a - a_1) \frown \mathsf{D}\,\Psi(\mathfrak{R})] \restriction \mathfrak{D} \dotplus (b - b_1) \restriction \mathfrak{D}. \\ \mathsf{D}\,\varphi(\Psi(\mathfrak{R})) = [a_1 b_1 \frown \mathsf{D}\,\Psi(\mathfrak{R}) \frown \mathsf{D}\,(\varphi\Psi)(\mathfrak{R})] \\ \qquad\qquad \dotplus\, [b_1(a - a_1) \frown \mathsf{D}\,\Psi(\mathfrak{R})] \dotplus (b - b_1). \end{cases}$$

13. 3. From the second term in (15) and (15a) we can see that the operators $\varphi(\Psi(\mathfrak{R}))$ and $\Psi(\varphi(\mathfrak{R}))$ may be not maximal normal. We also see that they differ. The operator $\varphi(\Psi(\mathfrak{R}))$ is acting in the spaces

$$a_1 b_1,\ a_1(b - b_1) \text{ and } (a - a_1),$$

which are disjoint with one another. The operator $\Psi(\varphi(\mathfrak{R}))$ is acting in the spaces

$$a_1 b_1,\ b_1(a - a_1) \text{ and } (b - b_1).$$

Later we shall show that the domains in these spaces are everywhere dense with respect to them.

Now we like to find the common operation-space for both $\varphi(\Psi(\mathfrak{R}))$ and $\Psi(\varphi(\mathfrak{R}))$. We shall examine the space

$$[a_1(b - b_1) + (a - a)] \cdot [b_1(a - a_1) + (b - b_1)].$$

A simple computation shows that this space coincides with $ab - a_1 b_1$. Consequently the operators $\Psi(\varphi(\mathfrak{R}))$ and $\varphi(\Psi(\mathfrak{R}))$ operate in the common space

$$a_1 b_1 + (ab - a_1 b_1) = ab.$$

13. 4. Let us see by what amount the operation-space of $\Psi(\varphi(\mathfrak{R}))$ exceeds the space ab. The difference is

$$a_1 b + (a - a_1) - ab = a - a_1 - b.$$

Thus outside of the common part with $\varphi(\Psi(\mathfrak{R}))$, the operation-space of $\Psi(\varphi(\mathfrak{R}))$ has $a - a_1 - b$ as an additional space.

Thus $\varphi(\Psi(\mathfrak{R}))$ and $\Psi(\varphi(\mathfrak{R}))$ have the common space ab and in addition to that, $\Psi(\varphi(\mathfrak{R}))$ has the additional space $a - a_1 - b$, and $\varphi(\Psi(\mathfrak{R}))$ has the additional space $b - b_1 - a$.

Of course, all three spaces ab, $a - a_1 - b$, $b - b_1 - a$ are disjoint.

In the additional spaces the operators are $a - a_1 - b \restriction \mathfrak{D}$ and $b - b_1 - a \restriction \mathfrak{D}$ respectively.

We see that concerning the common operation space $a \cdot b$, the said operators may differ in their domains. We have

$$(1)\quad \begin{cases} \Psi(\varphi(\mathfrak{R})) = a \cdot b \restriction \Psi(\varphi(\mathfrak{R})) \dotplus (a - a_1 - b) \restriction \mathfrak{D}, \\ \varphi(\Psi(\mathfrak{R})) = a \cdot b \restriction \varphi(\Psi(\mathfrak{R})) \dotplus (b - b_1 - a) \restriction \mathfrak{D}. \end{cases}$$

We shall find other formulas for the combined operators. Notice that $\boldsymbol{a} \cdot \boldsymbol{b}$ contains the part $(\boldsymbol{a} - \boldsymbol{a}_1)\,\boldsymbol{b}$ where the operator $\Psi(\varphi(\mathfrak{R}))$ is null in the whole space $(\boldsymbol{a} - \boldsymbol{a}_1)\,\boldsymbol{b}$. Indeed the operator is null on the whole space $\boldsymbol{a} - \boldsymbol{a}_1$, (15) in [13. 2. 2], hence also in $(\boldsymbol{a} - \boldsymbol{a}_1)\,\boldsymbol{b}$.

In the remainder $\boldsymbol{ab} - (\boldsymbol{a} - \boldsymbol{a}_1)\,\boldsymbol{b} = [\boldsymbol{a} - (\boldsymbol{a} - \boldsymbol{a}_1)]\,\boldsymbol{b} = \boldsymbol{a}_1\boldsymbol{b}$, the operator $\Psi(\varphi(\mathfrak{R}))$ may be split into two, working in the terms, (spaces), of:

$$\boldsymbol{a}_1\boldsymbol{b}_1 + \boldsymbol{a}_1(\boldsymbol{b} - \boldsymbol{b}_1) = \boldsymbol{a}_1\boldsymbol{b}.$$

In $\boldsymbol{a}_1(\boldsymbol{b} - \boldsymbol{b}_1)$ it is a null operator, but not necessarily on the whole space, but (see (15) in [13. 2. 2]) on the everywhere dense set

$$\mathfrak{D}\,\varphi(\mathfrak{R}) \cap \boldsymbol{a}_1(\boldsymbol{b} - \boldsymbol{b}_1).$$

Thus we can say that

$$(2) \qquad \begin{cases} \Psi(\varphi(\mathfrak{R})) = \boldsymbol{a}_1\boldsymbol{b} \restriction \Psi(\varphi(\mathfrak{R})) + (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{D}, \\ \varphi(\Psi(\mathfrak{R})) = \boldsymbol{ab}_1 \restriction \varphi(\Psi(\mathfrak{R})) + (\boldsymbol{b} - \boldsymbol{b}_1) \restriction \mathfrak{D}, \end{cases}$$

and we also can say that $\varphi(\Psi(\mathfrak{R}))$ can be split into two parts, the first of which being $\boldsymbol{a}_1\boldsymbol{b}_1 \restriction \Psi(\varphi(\mathfrak{R}))$, and the second is a null operator with domain everywhere dense in $\boldsymbol{a}_1(\boldsymbol{b} - \boldsymbol{b}_1) + (\boldsymbol{a} - \boldsymbol{a}_1)$.

Similar results we have for $\varphi(\Psi(\mathfrak{R}))$.

13. 5. We shall prove that our combined operators have their domains everywhere dense in the various spaces of their decomposition.

Lemma. *If* 1. $f(\dot{p})$ *is a* μ*-square summable function on the set* a,

2. $a \geqq a_1 \geqq a_2 \geqq \cdots$ *with* $\mu(a_n) \to 0$,

3. $f_n(p) \underset{\mathrm{df}}{=} \begin{cases} f(p) & \text{on } a - a_n \\ 0 & \text{on } a_n, \end{cases}$

then

$$\int_a |f_n(p) - f(p)|^2\, d\mu \to 0.$$

Proof.

$$\int_a |f_n(p) - f(p)|^2\, d\mu = \int_{a - a_n} 0\, d\mu + \int_{a_n} |f(p)|^2\, d\mu \to 0$$

because $\mu(a_n) \to 0$. Indeed the integral is a denumerably additive set function, which is μ-continuous.

Having this lemma, suppose that the set $A_1 B_1$ has a positive measure, and consider the functions $\varphi(z)$, $\Psi(z)$, restricted to $A_1 \cdot B_1$. Put

$$E_n \underset{\mathrm{df}}{=} \{z \mid n - 1 \leqq |\varphi(z)| < n\}$$

$$F_n \underset{\mathrm{df}}{=} \{z \mid n - 1 \leqq |\Psi(z)| < n\} \text{ for } n = 1, 2, \ldots.$$

Since the sets E_n are disjoint and $\sum_{n=1}^{\infty} E_n = A_1 B_1$, it follows that $\mu\left(\sum_{n=k}^{\infty} E_n\right) \to 0$ for $k \to \infty$. Similarly we have $\mu\left(\sum_{n=k}^{\infty} F_n\right) \to 0$ for $k \to \infty$. It follows that $\mu D_k \to 0$ where $D_k \underset{\mathrm{df}}{=} \sum_{n \geqq k \text{ or } m \geqq k} E_n F_m$.

Take any μ-square summable function $Y(\dot{p})$ defined μ-a. e. on $A_1' B_1'$. Define

$$Y_k(\dot{p}) \underset{\mathrm{df}}{=} \begin{cases} Y(\dot{p}) & \text{on } \mathrm{co}_V D_k' \\ 0 & \text{on } D_k'. \end{cases}$$

The functions

$$\varphi(\mathrm{Num}\,\dot{p})\cdot Y(\dot{p})$$
$$\Psi(\mathrm{Num}\,\dot{p})\cdot\varphi(\mathrm{Num}\,\dot{p})\cdot Y_k(\dot{p})$$

if restricted to $A_1^{\cdot}B_1^{\cdot}$, are μ-square summable, hence the $\mathfrak{C}^{-1}$-image $\vec{Y}_k$ of $Y_k(\dot{p})$ belongs (by (15) in [13. 2. 2]) to

$$\mathsf{D}\,[\boldsymbol{a}_1\boldsymbol{b}_1 \restriction \Psi(\varphi(\mathfrak{N}))] = \boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,\varphi(\mathfrak{N}) \frown \mathsf{D}\,(\varphi\Psi)\,\mathfrak{N}.$$

On the other hand, by the lemma, the functions $Y_k(\dot{p})$, if we vary $Y(\dot{p})$, make an everywhere dense set in $\mathcal{H}^V(A_1^{\cdot}B_1^{\cdot})$. Consequently the set of vectors

$$\boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,\varphi(\mathfrak{N}) \frown \mathsf{D}\,(\varphi\Psi)\,\mathfrak{N}$$

is everywhere dense in $\boldsymbol{a}_1\boldsymbol{b}_1$. Similarly we prove that

$$\boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,\Psi(\mathfrak{N}) \frown \mathsf{D}\,(\Psi\varphi)\,\mathfrak{N}$$

is everywhere dense in $\boldsymbol{a}_1\boldsymbol{b}_1$.

Consequently in the decomposition formulas (15), (15a) the domains in all three decomposition spaces are everywhere dense.

The above will be general if we agree to consider the set composed of the single vector $\vec{O}$ as everywhere dense in the space O.

13. 6. Consider the decomposition formula (15) in [13. 2. 2]:

$$(1)\quad \Psi(\varphi(\mathfrak{N})) = [\boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,\varphi(\mathfrak{N}) \frown \mathsf{D}\,(\Psi\varphi)\,(\mathfrak{N})] \restriction (\Psi\varphi)\,(\mathfrak{N})$$
$$+ [\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1) \frown \mathsf{D}\,\varphi(\mathfrak{N})] \restriction \mathfrak{O} \dotplus (\boldsymbol{a}-\boldsymbol{a}_1) \restriction \mathfrak{O}.$$

Since the operator $(\Psi\varphi)\,\mathfrak{N}$ restricted to $\boldsymbol{a}_1\boldsymbol{b}_1$ is maximal normal in $\boldsymbol{a}_1\boldsymbol{b}_1$, the domain $\mathsf{D}\,(\varphi\Psi)\,(\mathfrak{N}) \frown \boldsymbol{a}_1\boldsymbol{b}_1$ is everywhere dense in $\boldsymbol{a}_1\boldsymbol{b}_1$. Since, by [13. 5] the set $\boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,\varphi(\mathfrak{N}) \frown \mathsf{D}\,(\Psi\varphi)\,(\mathfrak{N})$ is everywhere dense in $(\boldsymbol{a}_1\boldsymbol{b}_1)$ and contained in $\boldsymbol{a}_1\boldsymbol{b}_1 \frown \mathsf{D}\,(\varphi\Psi)\,(\mathfrak{N})$, it follows that the operator

$$\boldsymbol{a}_1\boldsymbol{b}_1 \restriction (\Psi\varphi)\,(\mathfrak{N})$$

is a maximal normal extension in $\boldsymbol{a}_1\boldsymbol{b}_1$ of the first term in (1). Indeed

$$\mathsf{D}\,\varphi(\mathfrak{N}) \frown \mathsf{D}\,(\Psi\varphi)\,(\mathfrak{N}) \frown \boldsymbol{a}_1\boldsymbol{b}_1$$

is an everywhere dense part of $\mathsf{D}\,(\Psi\varphi)\,(\mathfrak{N}) \frown \boldsymbol{a}_1\boldsymbol{b}_1$. Since $\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1) \frown \mathsf{D}\,\varphi(\mathfrak{N})$ is everywhere dense in $\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1)$, the operator $\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1) \restriction \mathfrak{O}$ is a maximal normal extension in $\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1)$, of the second term of (1). Thus though $\Psi(\varphi(\mathfrak{N}))$ may not be a maximal operator, it has a maximal normal extension in its operation-space, viz.

$$\boldsymbol{a}_1\boldsymbol{b}_1 \restriction (\Psi\varphi)\,\mathfrak{N} \dotplus \boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1) \restriction \mathfrak{O} \dotplus (\boldsymbol{a}-\boldsymbol{a}_1) \restriction \mathfrak{O},$$

i. e.

$$(\boldsymbol{a}_1\boldsymbol{b}_1) \restriction (\Psi\varphi)\,(\mathfrak{N}) \dotplus [\boldsymbol{a}_1(\boldsymbol{b}-\boldsymbol{b}_1) + (\boldsymbol{a}-\boldsymbol{a}_1)] \restriction \mathfrak{O}.$$

Similarly the operator $\varphi(\Psi(\mathfrak{N}))$ has a maximal normal extension

$$(\boldsymbol{a}_1\boldsymbol{b}_1) \restriction (\varphi\Psi)\,(\mathfrak{N}) \dotplus [\boldsymbol{b}_1(\boldsymbol{a}-\boldsymbol{a}_1) + (\boldsymbol{b}-\boldsymbol{b}_1)] \restriction \mathfrak{O}.$$

We see that even these maximal normal extensions are not identical. They have only a common part which may be not trivial: $(\boldsymbol{a}_1\boldsymbol{b}_1) \restriction (\Psi\varphi)\,(\mathfrak{N})$. Concerning the operator $(\Psi\varphi)\,(\mathfrak{N})$, we have, by [11]

$$(\Psi\varphi)\,(\mathfrak{N}) = \boldsymbol{a}_1\boldsymbol{b}_1 \restriction (\Psi\varphi)\,(\mathfrak{N}) \dotplus (\boldsymbol{a}\boldsymbol{b}-\boldsymbol{a}_1\boldsymbol{b}_1) \restriction \mathfrak{O}.$$

Hence, the mentioned normal maximal extensions of all three operators $\Psi(\varphi(\mathfrak{N}))$, $\varphi(\Psi(\mathfrak{N}))$, $(\varphi\Psi)\,(\mathfrak{N})$ differ, though they have a mattering common part.

13. 7. Remark. In this paper we do not consider the problem of uniqueness of the maximal normal extension of the combined operators.

13. 8. Let us consider some special cases.

1) Case: $A = A_1$, $B = B_1$:

The decomposition formulas are reduced to

$$\Psi(\varphi(\mathfrak{N})) = [\boldsymbol{a}_1 \boldsymbol{b}_1 \cap \square \varphi(\mathfrak{N})] \restriction (\Psi\varphi)(\mathfrak{N}),$$
$$\varphi(\Psi(\mathfrak{N})) = [\boldsymbol{a}_1 \boldsymbol{b}_1 \cap \square \psi(\mathfrak{N})] \restriction (\Psi\varphi)(\mathfrak{N}).$$

Their common maximal normal extension in $\boldsymbol{a}_1 \boldsymbol{b}_1$ is $(\boldsymbol{a}_1 \boldsymbol{b}_1) \restriction (\Psi\varphi)(\mathfrak{N})$ and this is (by 11) the operator $(\Psi\varphi)(\mathfrak{N})$. The operators are in $\boldsymbol{a}_1 \boldsymbol{b}_1$.

2) Case: $A = B$, $A_1 = B_1$:

The decomposition formulas are:

$$\Psi(\varphi(\mathfrak{N})) = [\boldsymbol{a}_1 \cap \square \varphi(\mathfrak{N})] \restriction (\Psi\varphi)(\mathfrak{N}) \dot{+} (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{O},$$
$$\varphi(\Psi(\mathfrak{N})) = [\boldsymbol{a}_1 \cap \square \Psi(\mathfrak{N})] \restriction (\varphi\Psi)(\mathfrak{N}) \dot{+} (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{O}.$$

The common maximal normal extension is

$$\boldsymbol{a}_1 \restriction (\Psi\varphi)(\mathfrak{N}) \dot{+} (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{O}$$

which coincides with the operator $(\Psi\varphi)(\mathfrak{N})$. It operates in the space $\boldsymbol{a}$.

3) Case: $A = A_1 = B = B_1$: We have

$$\Psi(\varphi(\mathfrak{N})) = [\boldsymbol{a} \cap \square \varphi(\mathfrak{N})] \restriction (\Psi\varphi)\mathfrak{N},$$
$$\varphi(\Psi(\mathfrak{N})) = [\boldsymbol{a} \cap \square \Psi(\mathfrak{N})] \restriction (\Psi\varphi)\mathfrak{N}.$$

The operator $(\varphi\Psi)\mathfrak{N}$ is their common maximal normal eytension in $\boldsymbol{a}$. The operators are acting in $\boldsymbol{a}$.

14. Inverse operator. Let $\varphi(z)$ be defined on A only, and let $\varphi(z) \neq 0$ on A_1 where $A_1 \subseteq A$. Both subsets A, A_1 of the plane P are supposed to be N-measurable, as well as $\varphi(z)$.

The function $\Psi(z) \underset{\mathrm{df}}{=} \dfrac{1}{\varphi(z)}$ is defined on A_1 only, and we have $\Psi(z) \neq 0$ on A_1.

By [11] we have $\square \varphi(\mathfrak{N}) = \square \varphi_1(\mathfrak{N}) + (\boldsymbol{a} - \boldsymbol{a}_1)$ where

$$\varphi_1(z) = \begin{cases} \varphi(z) & \text{for } z \in A_1, \\ 0 & \text{for } z \in A - A_1 \end{cases}, \text{ and } \boldsymbol{a} \underset{\mathrm{df}}{=} \hat{\mathfrak{C}}^{-1} \mathcal{H}^V(A^{\cdot}),\ \boldsymbol{a}_1 \underset{\mathrm{df}}{=} \hat{\mathfrak{C}}^{-1} \mathcal{H}^V(A_1^{\cdot}).$$

We also have $\varphi_1(\mathfrak{N}) = \boldsymbol{a}_1 \restriction \varphi(\mathfrak{N})$ and

$$\varphi(\mathfrak{N}) = \varphi_1(\mathfrak{N}) \dot{+} (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{O}.$$

For $\Psi(\mathfrak{N})$ we have:

$$\frac{1}{\varphi}(\mathfrak{N}) = \boldsymbol{a}_1 \restriction \frac{1}{\varphi}(\mathfrak{N}).$$

If $\boldsymbol{a}_1 = \boldsymbol{0}$ then $\dfrac{1}{\varphi}(\mathfrak{N}) = \mathfrak{O}$ in O.

The general decomposition formulas give the following:

$$\frac{1}{\varphi}(\varphi(\mathfrak{N})) = [\boldsymbol{a}_1 \cap \square \varphi(\mathfrak{N})] \restriction \mathfrak{J} \dot{+} (\boldsymbol{a} - \boldsymbol{a}_1) \restriction \mathfrak{O}$$

$$\varphi\left(\frac{1}{\varphi}(\mathfrak{N})\right) = \left[\boldsymbol{a}_1 \cap \square \frac{1}{\varphi}(\mathfrak{N})\right] \restriction \mathfrak{J}$$

where $\mathfrak{J}$ is the "identity" operator, i. e. carrying vectors into themselves. The operator

$$a_1 \restriction \mathfrak{J} + (a - a_1) \restriction \mathfrak{D}$$

is a maximal normal extension of $\frac{1}{\varphi}(\varphi(\mathfrak{N}))$, and $a_1 \restriction \mathfrak{J}$ is the maximal normal extension of $\varphi\left(\frac{1}{\varphi}(\mathfrak{N})\right)$. The operator $a_1 \restriction \mathfrak{J}$ is identical with $\left(\varphi \cdot \frac{1}{\varphi}\right)(\mathfrak{N})$. A necessary and sufficient condition for the operator $\varphi(\mathfrak{N})$ to possess the inverse $[\varphi(\mathfrak{N})]^{\smile}$ is:

$$\text{“}\varphi(\mathfrak{N})\, \vec{X} = \vec{0} \text{ implies } \vec{X} = \vec{0}\text{”}.$$

If we take the $\mathfrak{C}$-image, we get

$$[\varphi(\mathfrak{N})]^* X(\dot{p}) = 0 \text{ implies } X(p) =^{\mu} 0.$$

Hence for vectors in $\mathcal{H}^V(A^{\cdot})$, the condition is:

$$\text{“}\varphi^P(\text{Num } p) \cdot X(\dot{p}) = 0 \text{ implies } X(\dot{p}) =^{\mu} 0\text{”}.$$

Hence if $\mu(A - A_1) > 0$, the inverse $[\varphi(\mathfrak{N})]^{\smile}$ does not exist, though $\frac{1}{\varphi}(\mathfrak{N})$ has a meaning.

The operator $a_1 \restriction \varphi(\mathfrak{N})$ possesses the inverse: $a_1 \restriction \frac{1}{\varphi}(\mathfrak{N}) = \frac{1}{\varphi}(\mathfrak{N})$. If $a = a_1 \neq 0$, then $\frac{1}{\varphi}(\mathfrak{N})$ is the inverse of $\varphi(\mathfrak{N})$.

In this case we get: $\varphi\left(\frac{1}{\varphi}(\mathfrak{N})\right)$ is the operation $\mathfrak{J}$ restricted to the range of $\varphi(\mathfrak{N})$, and $\frac{1}{\varphi}(\varphi(\mathfrak{N}))$ is the operation $\mathfrak{J}$ restricted to the domain of $\varphi(\mathfrak{N})$.

14a. Remark. The operational calculus does not depend on the choice of the canonical mapping, hence it is independent of the saturating sequence $\{\mathfrak{H}_n\}$ of spaces, whose main practical aim is to split multiple spectrum (1). Since such a splitting does not matter in the present paper, we may ponder about the possibility of simplifying our general theory, just by considering directly the borelian spectral tribe only, (which may be not saturated). Such modified approach however, will not change the vectors of $\mathfrak{H}$ into functions, but, we would have a mapping of functionoids based on a tribe of spaces into functionoids based on a tribe of subsets of the plane. The reader interested in such a possibility, may consult (3). We must however, add that our admitted approach is useful too, for it will give a more natural and finer classification of spectrum, and a better analysis of the normal operator than the usual one.

15. Resolvent of a maximal normal operator. We shall see in what simple and natural way our theory yields general results. We remind that, as in foregoing subsections, equality means both μ-equality in V and N-equality in P. Let $\alpha \in P$; (α) the set composed of the single complex number α. We put $a \underset{\mathrm{df}}{=} \mathcal{H}^V((\alpha)^{\cdot})$ and $a \underset{\mathrm{df}}{=} \hat{\mathfrak{C}}^{-1} a$. It may happen that $\mu((\alpha)) > 0$.

We put

$$\varphi(z) \underset{\mathrm{df}}{=} \dot{z} - \alpha \text{ and } \Psi(\dot{z}) \underset{\mathrm{df}}{=} \frac{1}{\dot{z} - \alpha},$$

and shall study the operators $\varphi(\mathfrak{N})$ and $\Psi(\mathfrak{N})$. There are two cases:

1) $\mu((\alpha)) > 0$, 2) $\mu((\alpha)) = 0$.

15. 1. Consider the first case: $\mu((\alpha)) > 0$. The function $\Psi(\dot{z})$ is not defined for $z = \alpha$. Put $P_1 \underset{\mathrm{df}}{=} P - (\alpha)$, and $\varphi_1(\dot{z}) \underset{\mathrm{df}}{=} P_1 \restriction \varphi(\dot{z})$. We have $(\alpha) = P - P_1$ co $(\alpha) = P_1$, $\varphi_1(z) \neq 0$ on P_1, and co $a = \hat{\mathfrak{C}}^{-1} \mathcal{H}^V(P_1^{\cdot})$.

Referring to [14] we get:

$$\varphi_1(\mathfrak{N}) = \mathrm{co}\, \boldsymbol{a} \restriction \varphi(\mathfrak{N}), \quad \varphi(\mathfrak{N}) = \varphi_1(\mathfrak{N}) \dot{+} \boldsymbol{a} \restriction \mathfrak{D},$$

$\frac{1}{\varphi}(\mathfrak{N}) = \Psi(\mathfrak{N}) = \mathrm{co}\, \boldsymbol{a} \restriction \frac{1}{\varphi_1}(\mathfrak{N})$; this is the operator inverse to $\varphi_1(\mathfrak{N}) = \mathrm{co}\, \boldsymbol{a} \restriction \varphi(\mathfrak{N})$, i. e. $[\varphi_1(\mathfrak{N})]^{\smile}$. We also have

$$\mathsf{D}\, \varphi(\mathfrak{N}) = \mathsf{D}\, \varphi_1(\mathfrak{N}) \dot{+} \boldsymbol{a} = \mathsf{D}\, [\mathrm{co}\, \boldsymbol{a} \restriction \varphi(\mathfrak{N})] \dot{+} \boldsymbol{a} = [\mathrm{co}\, \boldsymbol{a} \cap \mathsf{D}\, \varphi(\mathfrak{N})] \dot{+} \boldsymbol{a}.$$

$$\mathsf{D}\Psi(\mathfrak{N}) = \mathsf{D}\left[\mathrm{co} \restriction \frac{1}{\varphi}(\mathfrak{N})\right] = \mathrm{co}\, \boldsymbol{a} \cap \mathsf{D}\, \frac{1}{\varphi}(\mathfrak{N}) = \mathsf{D}\, \frac{1}{\varphi}(\mathfrak{N}).$$

Using the commonly admitted symbolism, we can write

$$\varphi(\mathfrak{N}) = \mathfrak{N} - \alpha\mathfrak{J} = \mathrm{co}\, \boldsymbol{a} \restriction (\mathfrak{N} - \alpha\mathfrak{J}) \dot{+} \boldsymbol{a} \restriction \mathfrak{D}$$

$$\Psi(\mathfrak{N}) = \frac{1}{\mathfrak{N} - \alpha\mathfrak{J}} = \mathrm{co}\, \boldsymbol{a} \restriction \frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$$

$$\mathsf{D}\, \varphi(\mathfrak{N})\, [\mathrm{co}\, \boldsymbol{a} \cap \mathsf{D}\, \mathfrak{N}] \dot{+} \boldsymbol{a} = \text{domain of } \mathrm{co}\, \boldsymbol{a} \restriction \mathfrak{N} \dot{+} \boldsymbol{a},$$

$$\mathsf{D}\Psi(\mathfrak{N}) = \text{range of } \mathrm{co}\, \boldsymbol{a} \restriction \mathfrak{N}.$$

We also have $\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}(\mathfrak{N} - \alpha\mathfrak{J}) = \mathfrak{J}$ restricted to the domain of $\mathrm{co}\, \boldsymbol{a} \restriction (\mathfrak{N}) \dot{+} \boldsymbol{a} \restriction \mathfrak{D}$, $(\mathfrak{N} - \alpha\mathfrak{J})\left(\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\right) = \mathfrak{J}$ restricted to the range of $\mathrm{co}\, \boldsymbol{a} \restriction (\mathfrak{N} - a\mathfrak{J})$. The domain of $\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}(\mathfrak{N} - \alpha\mathfrak{J})$ is $\mathsf{D}\, (\mathrm{co}\, \boldsymbol{a} \restriction \mathfrak{N}) \dot{+} a$.

The domain of $(\mathfrak{N} - \alpha\mathfrak{J})\left(\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\right)$ is the range of $\mathrm{co}\, \boldsymbol{a} \restriction (\mathfrak{N} - \alpha\mathfrak{J})$. The operator $\mathfrak{N} - \alpha\mathfrak{J}$ has no inverse, but $\mathrm{co}\, \boldsymbol{a} \restriction (\mathfrak{N} - \alpha\mathfrak{J})$ has one, viz. $\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$.

If we put $\mathfrak{N}' \underset{\mathrm{df}}{=} \mathrm{co} \restriction (\mathfrak{N} - \alpha\mathfrak{J})$ we have

$$\mathfrak{N}'(\mathfrak{N}'^{\smile}) = (\text{Range of } \mathfrak{N}') \restriction \mathfrak{J},$$

$$\mathfrak{N}'^{\smile}(\mathfrak{N}) = (\text{Domain of } \mathfrak{N}') \restriction \mathfrak{J}.$$

15. 2. Consider the second case $\mu((\alpha)) = 0$. Then, with respect to μ-equality, we have from [14], that $\frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ is the inverse operator of the operator $\mathfrak{N} - \alpha\mathfrak{J}$.

The domain of the first is the range of $\mathfrak{N} - \alpha\mathfrak{J}$.

The domain of the second is that of $\mathfrak{N}$. Both are everywhere dense in $\mathfrak{H}$.

The above results are general.

15. 3. We shall proceed to the spectral analysis of the given operator $\mathfrak{N}$. We know that it is not the measure which matters, but the ideal $\mathfrak{J}_N$ of $\mathfrak{N}$-null sets on P. The cases considered in 15. 1 and 15. 2 are those where $(\alpha) \bar{\in} \mathfrak{J}_N$ and $(\alpha) \in \mathfrak{J}_N$ respectively.

Various possibilities which we are now going to consider in relation to $\mathfrak{J}_N$ may be termed *global properties of the operator* $\mathfrak{N}$. (See (4), p. 98). The point α will be termed *globally elusive and regular*[13]), whenever there exists $r > 0$ such that the set

$$\{z \mid |z - \alpha| < r\} \in \mathfrak{J}_N.$$

It will be termed *globally elusive and singular*, whenever $(\alpha) \in \mathfrak{J}_N$, but for every $r > 0$

$$\{z \mid |z - \alpha| < r\} \bar{\in} \mathfrak{J}_N;$$

[13]) Called by Stone (2): "point of the resolvent set".

the point α will be termed *globally heavy and regular* whenever $(\alpha) \bar{\in} \mathfrak{J}_N$, but there exists $r > 0$ such that

$$\{z \mid 0 < |z - \alpha| < r\} \in \mathfrak{J}_N.$$

It will be termed *globally heavy and singular* whenever $(\alpha) \bar{\in} \mathfrak{J}_N$ and for every $r > 0$

$$\{z \mid 0 < |z - \alpha| < r\} \bar{\in} \mathfrak{J}_N.$$

We shall study these four possibilities separately.

15. 4. First we shall study the property of being globally elusive and regular. The set (GER) of all points α of this category is an open set on the plane P. The set may be empty or even be equal P. The complementary to (GER) is called the *global spectrum of* $\mathfrak{N}$.

Let $\alpha \in (GER)$. Find $r > 0$ such that

$$R \underset{\mathrm{df}}{=} \{z \mid |z| < r\} \in \mathfrak{J}_N.$$

We have

$$\varphi(z) \underset{\mathrm{df}}{=} \frac{1}{z - \alpha} =^N \left((P - R) \upharpoonright \frac{1}{z - \alpha}\right) \underset{\mathrm{df}}{=} \Psi(z).$$

The function $\Psi(\mathrm{Num}\, \dot{p})$ is bounded on V. Let $|\Psi(\mathrm{Num}\, p)| \leqq M$. We get

$$\int_V |\Psi(N)\, X(\dot{p})|^2\, d\mu < M^2 \int_V |X(\dot{p})|^2\, d\mu.$$

Hence, taking $\mathfrak{C}^{-1}$ images we get $\|\varphi(\mathfrak{N})\, \vec{X}\| < \|\vec{X}\| \cdot M$ for all $\vec{X} \in \mathfrak{H}$.

Consequently $\varphi(\mathfrak{N})$ is a bounded, hence continuous operator with domain $\mathfrak{H}$. We have proved that

Theorem. *If α is a globally elusive regular point, then $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ is the inverse operator to $\mathfrak{N} - \alpha\mathfrak{J}$. It is bounded, continuous and defined over all $\mathfrak{H}$.*

15. 5. Let α be globally elusive and singular. In this case $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ is also the inverse of $\mathfrak{N} - \alpha\mathfrak{J}$. The domain and range of $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ are both everywhere dense in $\mathfrak{H}$. Both are maximal normal operators. We shall prove that there exists a μ-square summable function $f(\dot{p})$ defined μ-a. e. on V such that $\dfrac{1}{\mathrm{Num}\, \dot{p} - \alpha} f(\dot{p})$ is not μ-square summable on V.

Denote by $k(r)$ the open circle with radius $r > 0$ and with center at α. Put $b_0 \underset{\mathrm{df}}{=} 1$. We have

(1) $$\mu k(1) > 0.$$

There exists b_1 such that $0 < b_1 < \dfrac{1}{\sqrt{2}}\, b_0$ and

(2) $$\mu\,[k(b_0) - k(b_1)] > 0.$$

Indeed the supposition that b_1 does not exist, would yield the following consequence: $\mu\left[k(b_0) - k\left(\dfrac{b_0}{n\sqrt{2}}\right)\right] = 0$ for all $n = 2, 3, \ldots$

Hence

$$\mu \sum_{n=2}^{\infty} \left[k(b_0) - k\left(\frac{b_0}{n\sqrt{2}}\right)\right] = 0.$$

Hence, since $\mu((\alpha)) = 0$, we would get $\mu k(b_0) = 0$ which contradicts (1).

Having that, repeat our argument, finding b_2 such that $0 < b_2 < \frac{1}{\sqrt{2}} b_1$,

$$\mu[k(b_1) - k(b_2)] > 0. \tag{3}$$

In general we shall have a number b_n such that

$$0 < b_n < \frac{1}{\sqrt{2}} b_{n-1}, \quad (n = 2, 3, \ldots),$$

$$\mu[k(b_{n-1}) - k(b_n)] > 0.$$

As

$$\mu[k(b_{n-1}) - k(b_n)] = \mu[k(b_{n-1}) - k(b_n)]^{\cdot} = \sum_{i=1}^{\infty} \mu[k(b_{n-1}) - k(b_n)]^{(i)} > 0,$$

where (i) refers to the i-th copy, which is a subset of V_n, there exists an i_n such that the i_n-th copy of $k(b_{n-1}) - k(b_n)$ has a positive measure. Take such an i_n and put $L_n \underset{\mathrm{df}}{=} i$-th copy of $k(b_{n-1}) - k(b_n)$. The sets L_n are mutually disjoint. Put

$$m_n \underset{\mathrm{df}}{=} \mu L_n.$$

Let us define the function $f(\dot{p})$ in V as follows:

$$f(p) = \sqrt{\frac{b_{n-1}^2 - b_n^2}{m_n}} \text{ on } L_n \text{ and } f(p) = 0 \text{ elsewhere.}$$

We have

$$\int_{L_n} |f(p)|^2 \, d\mu = b_{n-1}^2 - b_n^2.$$

Consequently $\int_V (f(p))^2 \, d\mu = \sum_{n=2}^{\infty} (b_{n-1}^2 - b_n^2) = b_1^2$, so $f(p)$ is μ-square-summable on V. We have

$$\int_{L_n} \left| \frac{1}{\mathrm{Num}\, p - \alpha} f(p) \right|^2 d\mu = \frac{b_{n-1}^2 - b_n^2}{m_n} \int_{L_n} \frac{1}{|\mathrm{Num}\, p - \alpha|^2} \, d\mu,$$

since, if $z \in k(b_{n-1}) - k(b_n)$, we have $b_n \leqq z - \alpha < b_{n-1}$.

Hence $\frac{1}{b_{n-1}} < \frac{1}{z - \alpha} \leqq \frac{1}{b_n}$. Thus we get

$$\int_{L_n} \left| \frac{f(p)}{\mathrm{Num}\, p - \alpha} \right|^2 d\mu \geqq \frac{b_{n-1}^2 - b_n^2}{b_{n-1}^2} \geqq \frac{1}{2}. \tag{1}$$

From (1) it follows that the series $\sum_{n=2}^{\infty} \int_{L_n} \left| \frac{f(p)}{\mathrm{Num}\, p - \alpha} \right|^2 d\mu$ diverges. Consequently the function $\frac{1}{\mathrm{Num}\, \dot{p} - \alpha} \cdot f(\dot{p})$ is not μ-square summable.

15. 5a. The above shows that the domain of $\frac{1}{\mathfrak{R} - \alpha\mathfrak{J}}$ differs from $\mathfrak{H}$, though it is everywhere dense. It follows that $\frac{1}{\mathfrak{R} - \alpha\mathfrak{J}}$ is discontinuous (not bounded) in $\mathfrak{H}$. We have proved:

Theorem. *If α is a globally elusive and singular complex number, then $\frac{1}{\mathfrak{R} - \alpha\mathfrak{J}}$ is an inverse operator to $\mathfrak{R} - \alpha\mathfrak{J}$. It is maximal, normal in $\mathfrak{H}$ but discontinuous.*

15. 5b. The converse theorem to [15. 4] and [15. 5a] is true. Thus we have:

Theorem. *If α is a globally elusive complex number, then*

1) $\mathfrak{N} - \alpha\mathfrak{J}$ *admits the inverse operator* $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *which is maximal normal.*

2) *The following are equivalent:*

I. $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is continuous,*

II. α *is a globally elusive regular number.*

3) *The following are equivalent:*

I. $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is discontinuous,*

II. α *is a globally elusive singular number.*

15. 6. Let us go over to the case where α is a **globally heavy number.**

The results of [15] give the following:

Put $a \underset{\mathrm{df}}{=} \mathcal{H}^{V}(\text{Point } \alpha)$, $\boldsymbol{a} \underset{\mathrm{df}}{=} \mathfrak{C}^{-1} a$.

The operator $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ is the inverse operator to $(\text{co } \boldsymbol{a}) \restriction \mathfrak{N} - \alpha\mathfrak{J}$. It is a maximal normal operator in co $\boldsymbol{a}$.

If we apply to co $\boldsymbol{a} \restriction \mathfrak{N} - \alpha\mathfrak{J}$ the result in theorem [15. 5. b] we get:

Theorem. *If α is a globally heavy complex number, then*

1) $\mathfrak{N} - \alpha\mathfrak{J}$ *does not admit any inverse operator,*

2) $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is the inverse of* $\mathfrak{N} - \alpha\mathfrak{J}$ *restricted to* co $\boldsymbol{a}$, *where* $\boldsymbol{a} = \mathfrak{C}^{-1}\mathcal{H}^{V}(\text{Point } \alpha)$, *and where* $\boldsymbol{a} \neq 0$ *is independant on the canonical mapping.*

3) *The operator* $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is maximal normal in* co $\boldsymbol{a}$.

4) *The following are equivalent:*

I. $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is continuous (in* co $\boldsymbol{a}$*),*

II. α *is a globally heavy regular number.*

5) *The following are equivalent:*

I. $\dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$ *is discontinuous (in* co $\boldsymbol{a}$*),*

II. α *is a globally heavy singular number.*

15. 7. Individual spectrums. If the borelian spectral tribe of $\mathfrak{N}$ is not saturated, then there exists a saturating sequence of spaces $\{\mathfrak{H}_n\}$ which may be finite or infinite. We may call the spaces $\mathfrak{H}_n$ *saturation compartments.* The operator $\mathfrak{H}_n \restriction \mathfrak{N}$ has a simple spectrum only. The results obtained in [15—15. 6] can be applied to each individual $\mathfrak{H}_n \restriction \mathfrak{N}$, so we can have for complex number α and $\mathfrak{H}_n$ the four possibilities discussed above. This classification of points α is, of course, dependent on the choice of the saturating sequence $\{\mathfrak{H}_n\}$. Given a point α which is globally elusive and regular it will be so in each compartment $\mathfrak{H}_n$. The same will be when α is globally heavy and regular.Then α will be so in each compartment $\mathfrak{H}_n$. A different behaviour show globally singular points.

The following possibilities may take place (both if α is elusive and if α is heavy):

1) α is regular in all compartments $\mathfrak{H}_n$, though it is globally singular.

2) α is singular in all compartments $\mathfrak{H}_n$,

3) α is regular in some compartments $\mathfrak{H}_n$, and singular in other compartments.

In relation to this there is the possibility of defining the "multiplicity of various above behaviour". It suggests the problem of independence of these multiplicities of the choice of the saturating sequence. The choice of the generating vector does not influence the ideal $\mathfrak{J}_\mu$ of μ-null sets on V. We leave these problems to the reader.

16. The equation $\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y}$. The equation can be treated by relying on the obtained theorems on the resolvent of $\mathfrak{N}$. Nevertheless, to show the advantage of the canonical mapping we shall treat it independently.

The canonical transformation $\mathfrak{C}$ transforms this equation into

$$\mathfrak{N}^* f(\dot{p}) - \alpha f(\dot{p}) \cdot =^\mu \cdot g(\dot{p}) \text{ on } V, \tag{1}$$

where the μ-square summable function $g(\dot{p})$ defined μ-a. e. on V is given, and $f(\dot{p})$ is to be found.

The equation (1) is equivalent to

$$(\mathfrak{N}^* - \alpha\mathfrak{J}^*)\, f(\dot{p}) =^\mu g(\dot{p}), \tag{2}$$

where $\mathfrak{J}^*$ is the operation which make correspond functions to themselves.

Case 1). α is globally elusive. The $\mathfrak{C}$-image of $\mathfrak{N} - \alpha\mathfrak{J}$ is the operator

$$(\mathfrak{N}^* - \alpha\mathfrak{J}^*)\, X(\dot{p}) \cdot =^\mu \cdot (\operatorname{Num} \dot{p} - \alpha) \cdot X(\dot{p}).$$

Hence (2) can be written

$$(\operatorname{Num} p - \alpha) \cdot f(p) \cdot =^\mu \cdot g(p). \tag{3}$$

Since $\operatorname{Num} \dot{p} - \alpha \cdot \neq^\mu \cdot 0$, the equation (3) is equivalent to

$$f(\dot{p}) \cdot =^\mu \cdot \frac{g(\dot{p})}{\operatorname{Num} \dot{p} - \alpha}. \tag{4}$$

$f(\dot{p})$ is μ-square summable if and only if $\dfrac{g(\dot{p})}{\operatorname{Num} \dot{p} - \alpha}$ is so, i. e. $g(\dot{p})$ must belong to the domain of the operator which carries $X(\dot{p})$ into $\dfrac{X(\dot{p})}{\operatorname{Num} \dot{p} - \alpha}$.

This is necessary and sufficient. Taking the $\mathfrak{C}^{-1}$-image we get the

16. 1. Theorem. *If α is globally elusive (regular or singular), then the following are equivalent:*

I. *The equation $\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y}$ has for the given $\vec{Y}$ a solution $\vec{X}$.*

II. $Y \in \mathfrak{D}\, \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$.

The solution is unique. The set of the admissible vectors $\vec{Y}$ is everywhere dense in $\mathfrak{H}$. The solution is $\vec{X} = \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\vec{Y}$.

If α is globally elusive and regular, there exists $r > 0$ such that $\operatorname{Num} p - \alpha > r$ almost μ-everywhere on P. Hence from (4) we get $f(\dot{p}) \leqq^\mu g(\dot{p}) \cdot \dfrac{1}{r}$. Consequently $f(\dot{p})$ is μ-square summable if and only if $g(\dot{p})$ is so. Thus all $g(\dot{p})$ give solution of (4).

We have proved the following

16. 2. Theorem. *If α is globally elusive and regular, then the equation*

$$\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y} \tag{5}$$

admits a solution for every $\vec{Y}$. This solution is $\vec{X} = \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\vec{Y}$. The solution is unique.

If α is globally elusive and singular, then the equation does not admit for every $\vec{Y}$ a solution.

Case 2). α is globally heavy. We have μ (Point α) > 0. The equation $\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y}$ goes, through $\mathfrak{C}$, into the equation $\mathfrak{N}^* X(\dot{p}) - \alpha X(\dot{p}) \cdot =^{\mu} \cdot Y(\dot{p})$. This equation is equivalent to

$$(\text{Num } \dot{p} - \alpha) \cdot X(\dot{p}) \cdot =^{\mu} \cdot Y(\dot{p}). \tag{5a}$$

Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(n)} \ldots$ be the copies of the number α on $V_1, V_2, \ldots, V_n \ldots$ respectively. There exists at least one n for which

$$\mu(\alpha^{(n)}) > 0.$$

Denote by $\alpha'_1, \alpha'_2, \ldots$ all copies of α with positive measure and by $\alpha''_1, \alpha''_2, \ldots$ the remaining copies of α.

Suppose that the μ-square summable function $X(\dot{p})$ satisfies (5). There exists a μ-null subset E of V, such that (Num $p - \alpha) \cdot X(p) = Y(p)$ everywhere on $V - E$. As E is a μ-null set, no one of $\alpha'_1, \alpha'_2, \ldots$ belongs to E. Hence $\alpha'_m \in V - E$ for all m. We get

$$0 \cdot X(\alpha'_m) = Y(\alpha'_m). \text{ Hence } Y(\alpha'_m) = 0 \text{ for all } m.$$

Let $a \underset{\mathrm{df}}{=}$ Point (α). We have $a \cdot =^{\mu} \cdot \bigcup_m \alpha'_m$, since the set of all α''_m does not matter, being a μ-null set. Hence

$$a \mid g(p) \cdot =^{\mu} \cdot 0, \text{ and then } \mathrm{Proj}_a\, g(p) \cdot =^{\mu} \cdot 0.$$

Taking the $\mathfrak{C}^{-1}$-image, and putting $\boldsymbol{a} = \mathfrak{C}^{-1}\mathcal{H}^V(a)$ we get $\mathrm{Proj}_{\boldsymbol{a}}\, \vec{Y} = \vec{0}$.

We have proved the

16. 3. Theorem. *If* 1. *α is globally heavy,*

2. $\boldsymbol{a} \underset{\mathrm{df}}{=} \mathfrak{C}^{-1}\mathcal{H}^V$ (Point α),

3. *$\vec{X}$ is a solution of the equation*

$$\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y}, \tag{6}$$

then $\vec{Y}$ is orthogonal to $\boldsymbol{a}$.

Now suppose that a vector $\vec{Y}$ is orthogonal to $\boldsymbol{a}$, and suppose that $\vec{X}$ is a solution of (6). Put $X(\dot{p}) \underset{\mathrm{df}}{=} \mathfrak{C}\vec{X}$, $Y(\dot{p}) \underset{\mathrm{df}}{=} \mathfrak{C}\vec{Y}$.

By what has been proved $\vec{Y}$ is orthogonal to $\boldsymbol{a}$. Hence (Point α) $\mid Y(\dot{p}) \cdot =^{\mu} \cdot 0$. Hence

$$\bigcup_n^{(n)} \alpha \mid Y(\dot{p}) \cdot =^{\mu} \cdot 0. \tag{7}$$

Since $X(\dot{p})$ is a solution of the $\mathfrak{C}$-transform of (6), we have

$$(\text{Num } \dot{p} = \alpha)\, X(\dot{p}) \cdot =^{\mu} \cdot Y(\dot{p}). \tag{8}$$

If $p = \alpha'_m$, then $Y(p) = 0$; this by (7). Hence, for $p = \alpha'_m$ the equation

$$(\text{Num } \dot{p} - \alpha)\, X(\dot{p}) = Y(\dot{p}) \tag{9}$$

is satisfied, because of (7). For $p = \alpha''_m$, the equation (9) may be satisfied or not: it does not matter, because of (7).

Let us consider the points p such that $\operatorname{Num} p \neq \alpha$. We get

$$X(\dot{p}) \cdot =^{\mu} \cdot \frac{Y(\dot{p})}{\operatorname{Num} \dot{p} - \alpha} \quad \text{on} \quad V - \bigcup_{n=1}^{\infty} (\alpha^{(n)}).$$

Hence $\dfrac{Y(p)}{\operatorname{Num} \dot{p} - \alpha}$ is μ-square summable on $V - \bigcup_{n=1}^{\infty} (\alpha^{(n)})$. It follows that $Y(\dot{p})$ belongs to the domain of the operator which carries the function $g(\dot{p})$ into $\dfrac{g(\dot{p})}{\operatorname{Num} \dot{p} - \alpha}$ and has the maximal domain. Hence $\vec{Y} \in \mathsf{D} \dfrac{1}{\mathfrak{R} - \alpha \mathfrak{J}}$.

Thus we have proved:

16. 3. 1. Theorem. *If* 1. *α is globally heavy,*

2. *$\vec{Y}$ is orthogonal to $\boldsymbol{a} \underset{\mathrm{df}}{=} \mathfrak{C}^{-1} \mathcal{H}^{V}$ (Point α),*

3. *The equation $\mathfrak{R} \vec{X} - \alpha \vec{X} = \vec{Y}$ possesses a solution $\vec{X}$,*

then $\vec{Y} \in \mathsf{D} \dfrac{1}{\mathfrak{R} - \alpha \mathfrak{J}}$.

The set $\mathsf{D} \dfrac{1}{\mathfrak{R} - \alpha \mathfrak{J}}$ coincides with $\boldsymbol{a}$ whenever α is globally heavy and regular. It is everywhere dense in $\operatorname{co} \boldsymbol{a}$, but not coinciding with $\operatorname{co} \boldsymbol{a}$, whenever α is globally heavy and singular.

Now we shall prove:

16. 4. Theorem. *If* 1. *α is globally heavy,*

2. *$\vec{Y}$ is orthogonal to $\boldsymbol{a} = \mathfrak{C}^{-1} \mathcal{H}^{V}$ (Point α),*

3. $\vec{Y} \in \mathsf{D} \dfrac{1}{\mathfrak{R} - \alpha \mathfrak{J}}$;

4. *$\vec{Z}$ is any vector of $\boldsymbol{a}$,*

then

$$\vec{X} \underset{\mathrm{df}}{=} \vec{Z} + \frac{1}{\mathfrak{R} - \alpha \mathfrak{J}} \vec{Y} \tag{1}$$

is a solution of the equation

$$\mathfrak{R} \vec{X} - \alpha \vec{X} = \vec{Y}. \tag{2}$$

For a given $\vec{Y}$ the only solutions of (2) *are* (1) *where $\vec{Z} \in \boldsymbol{a}$.*

Proof. Consider $\mathfrak{C}$-images, as before, and put $Z(\dot{p}) \underset{\mathrm{df}}{=} \mathfrak{C}\vec{Z}$. We have

$$Z(\dot{p}) =^{\mu} 0 \text{ on } \mathfrak{C}\boldsymbol{a} = \text{Point } \alpha,$$

and

$$Y(\dot{p}) =^{\mu} 0 \text{ on } V\text{-Point } \alpha.$$

From (1) we have

$$X(\dot{p}) \cdot =^{\mu} \cdot \frac{1}{\operatorname{Num} \dot{p} - \alpha} \cdot Y(\dot{p}) + Z(\dot{p}).$$

Hence

$$X(\dot{p}) \cdot =^{\mu} \cdot \frac{1}{\operatorname{Num} \dot{p} - \alpha} \cdot Y(\dot{p}) \text{ on Point } \alpha,$$

and

$$X(\dot{p}) \cdot =^{\mu} \cdot Z(\dot{p}) \text{ on } V\text{-Point } \alpha.$$

Hence

$$X(\dot{p}) \cdot (\operatorname{Num} \dot{p} - \alpha) =^{\mu} \cdot Y(\dot{p}) \text{ on Point } \alpha$$

and

$$X(\dot{p}) \cdot (\operatorname{Num} \dot{p} - \alpha) =^{\mu} Y(\dot{p}) \text{ on } V\text{-Point } \alpha,$$

because

$$X(\dot{p}) =^{\mu} Z(\dot{p}) =^{\mu} 0 \text{ on } V\text{-Point } \alpha$$

and also

$$Y(\dot{p}) =^{\mu} 0 \text{ on } V\text{-Point } \alpha.$$

Thus, through $\mathfrak{C}^{-1}$ we get

$$(\mathfrak{N} - \mathfrak{J}\alpha)\ \vec{X} = \vec{Y}, \text{ i. e. } \vec{X} \text{ satisfies (2).}$$

Conversely if $\vec{X}$ satisfies (2), we get, taking $\mathfrak{C}$-images:

$$(\text{Num } \dot{p} - \alpha) \cdot X(\dot{p}) \cdot =^{\mu} \cdot Y(\dot{p}).$$

We get on V-Point α:

$$X(\dot{p}) \cdot =^{\mu} \cdot \frac{Y(\dot{p})}{\text{Num } \dot{p} - \alpha},$$

and putting $Z(\dot{p}) \underset{\mathrm{df}}{=} \text{Point } \alpha \restriction X(\dot{p})$, we get

$$X(\dot{p}) \cdot =^{\mu} \cdot \frac{Y(\dot{p})}{\text{Num } \dot{p} - \alpha} + Z(\dot{p});$$

hence

$$\vec{X} = \frac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\ \vec{Y} + \vec{Z} \text{ where } \vec{Z} \in \boldsymbol{a}.$$

We have proved the

16. 5. Theorem. 1. *If* I. *α is globally heavy,*

II. *$\vec{X}$ is a solution of the equation $\mathfrak{N}\vec{X} - \alpha\vec{X} = \vec{Y}$,*

then $\vec{Y}$ is orthogonal to $\boldsymbol{a}$ where $\boldsymbol{a} = \mathfrak{C}^{-1} H^{V}$ (Point α).

2. $\vec{Y} \in \mathsf{C}\ \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$.

3. *There exists $\vec{Z} \in \boldsymbol{a}$ such that $\vec{X} = \vec{Z} + \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}\ \vec{Y}$.*

4. *If α is globally heavy and regular, the following are equivalent:*

I. $\vec{Y} \in \text{co}\ \boldsymbol{a}$,

II. *the solution exists.*

5. *If α is globally heavy and singular, the following are equivalent*

I. $\vec{Y} \in \mathsf{C}\ \dfrac{1}{\mathfrak{N} - \alpha\mathfrak{J}}$,

II. *the solution exists.*

In this case not for all $\vec{Y}$ there exists a solution, but the set of these $\vec{Y}$ for which the solution exists is dense in co $\boldsymbol{a}$.

References.

(1) *Otton Martin Nikodým*, Contribution à la théorie des opérateurs normaux, maximaux dans l'espace de Hilbert-Hermite séparable et complet; Journal de Mathématiques pures et appliquées, **36**, (1957), 129—146.

(2) *Marshal H. Stone*, Linear transformations in Hilbert space. New York 1932.

(3) *Otton Martin Nikodým*, Sur les êtres fonctionoïdes. I. Une généralisation de la notion de fonctions; C. R. **226** (1948), 375. — II. Procédé de complétion asymptotique; C. R. **226** (1948), 458. — III. Echelle spectrale et intégration des êtres fonctionoïdes; C. R. **226** (1948), 541.

(4) *N. J. Achieser* and *J. M. Glasman*, Theorie der linearen Operatoren im Hilbert-Raum. Berlin 1954.

Eingegangen 2. Mai 1959.

Note on Linear Operators.

By *Edward O. Thorp*, Los Angeles, Cal.

Introduction. This note answers two questions. 1.) If X and Y are Banach spaces, with Y inseparable, does it ever happen that all the bounded operators from X into Y are compact? 2.) If X and Y are Banach spaces, consider the following subset of the space of bounded operators from X into Y with the uniform topology: the operators (compact operators) with both a) non-dense range in Y, and b) which send some non-zero vector into zero. Does this subset of operators (compact operators) ever have an interior point?

Notation. X and Y denote Banach spaces. $[X, Y]$ is the set of bounded operators from X into Y. $[X, Y]_c$ is the subspace of compact operators from X into Y. $l^p(S)$, $\infty > p \geqq 1$, is the set of scalarvalued functions on S, an arbitrary set, having at most a countable number of non-zero coordinates and such that the norm of the function x (where we denote $x(s)$, s in S, by x_s) is given by

$$\| x \| = \Big(\sum_{s \in S} | x_s |^p\Big)^{\frac{1}{p}} < \infty.$$

We define $l^\infty(S)$, $c_0(S)$ and $c(S)$ analogously [4]. If S is countable we omit the (S). With a few exceptions, explained when they occur, the notation is taken from [8].

Some known cases in which all bounded operators are compact.

(a) $[l^p, l^q]$, $\infty > p > q \geqq 1$; $[c_0, l^q]$, $[c, l^q]$, $\infty > q \geqq 1$. For $[l^p, l^q]$, $\infty > p > q > 1$ see [3, p. 700] and [7, thm. 1]. The other cases are contained in (b), below.

(b) $[X, l^1]$ where X is either quasi-reflexive [2, thm. 5. 1], or weakly complete [6, Cor. 4. 11], or *has* a separable conjugate space [6, Cor. 4. 11]. [Note: 4. 11 states "X a separable conjugate space . . ." This is evidently false as the example $X = l^1$ (the separable conjugate of c_0) shows. The proof of 4. 11 indicates the corrected form.]

(c) Either X or Y is finite dimensional.

(d) If all the operators in $[Y', X']$ are compact, then all the operators in $[X, Y]$ are compact [8].

All the Y in the above list are separable. Also the range of a compact operator is separable [1]. These facts motivated question 1.

The spaces $l^q(T)$, $1 \leqq q < \infty$, are not separable when T is uncountable, so the following theorem answers question 1 by showing that all the bounded operators may be compact even if Y is not separable.

Theorem 1. *All the bounded operators are compact in the following cases:*

(a) $[l^p(S), l^q(T)]$, $\infty > p > q \geqq 1$; $[c_0(S), l^q(T)]$, $[c(S), l^q(T)]$, $\infty > q \geqq 1$.

(b) $[X, l^1(T)]$ *where X is either quasi-reflexive or weakly complete or has a separable conjugate space.*

Proof. Let A be a bounded linear operator. For case (a), let $X(S)$ be the domain and $Y(T)$ the range. If (x_n) is a bounded sequence in $X(S)$, (x_n) has only a countable number of non-zero coordinates among the totality of its members, so the closed span of (x_n), written $\overline{sp}(x_n)$, is linearly isometric to $X(J)$. Similarly, $\overline{sp}(Ax_n)$ is isometric to $Y(L)$ where L is either finite or J. Since A is continuous and linear, $A\left(\overline{sp(x_n)}\right) \subset \overline{A\left(sp(x_n)\right)} = \overline{sp}\,A(x_n)$. Therefore we can consider the restriction of A as an operator $\tilde{A}$ on $\overline{sp(x_n)}$ into $\overline{sp}(Ax_n)$. But the list of known results, listed in section 2, shows $\tilde{A}$ is compact hence $(\tilde{A}x_n) = (Ax_n)$ has a convergent subsequence, proving A is compact.

The preceding argument applies verbatim to case (b) because each of the properties involved is inherited by closed subspaces. For quasi-reflexivity, note that a norm-closed subspace of a weakly complete space is weakly closed, hence weakly complete [5; ch. 6 thm. 22]. If X' is separable, use the Hahn-Banach extension theorem.

Theorem 2. *If $X = l^p(S)$, $Y = l^q(T)$, $\infty > p > q > 1$, with S and T uncountable, then every bounded linear operator A from X into Y has a non-dense range, a non-zero kernel, and is compact.*

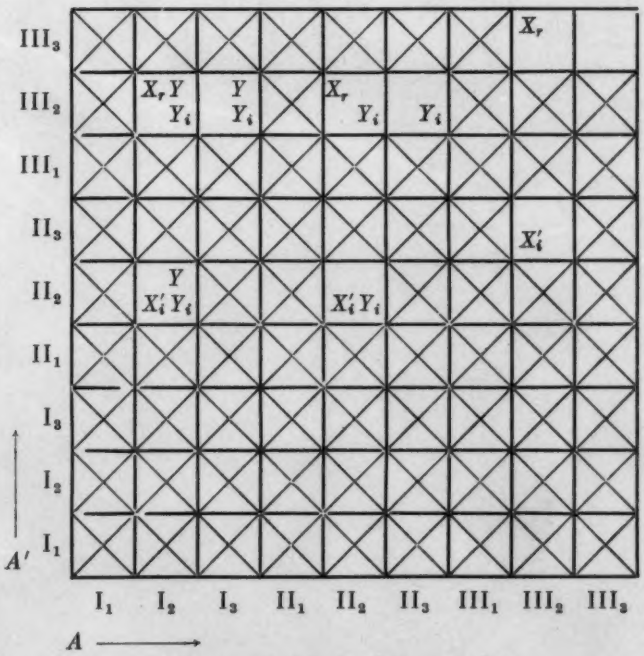
Proof. We will need the following lemmas.

Lemma 1. If A is compact, the range of A is separable [1].

Lemma 2. If A is compact and A has a bounded inverse, the dimensions of X and of the range of A are finite and equal.

The proof is immediate.

State Diagram for Compact Operators.



Y: impossible if Y is complete.
X_r: impossible if X is reflexive.
Y_i: impossible if Y is inseparable.
X_i': impossible if X' is inseparable.

Now refer to the "state diagram" of [8] (or [9]). Lemma 1 shows that states I and II are impossible for A when Y is not separable. Similarly, if X' is not separable, states I and II are impossible for A'. Lemma 2 says that if X and Y are both infinite dimensional (the assumption throughout this paper), state I is impossible for either A or A'.

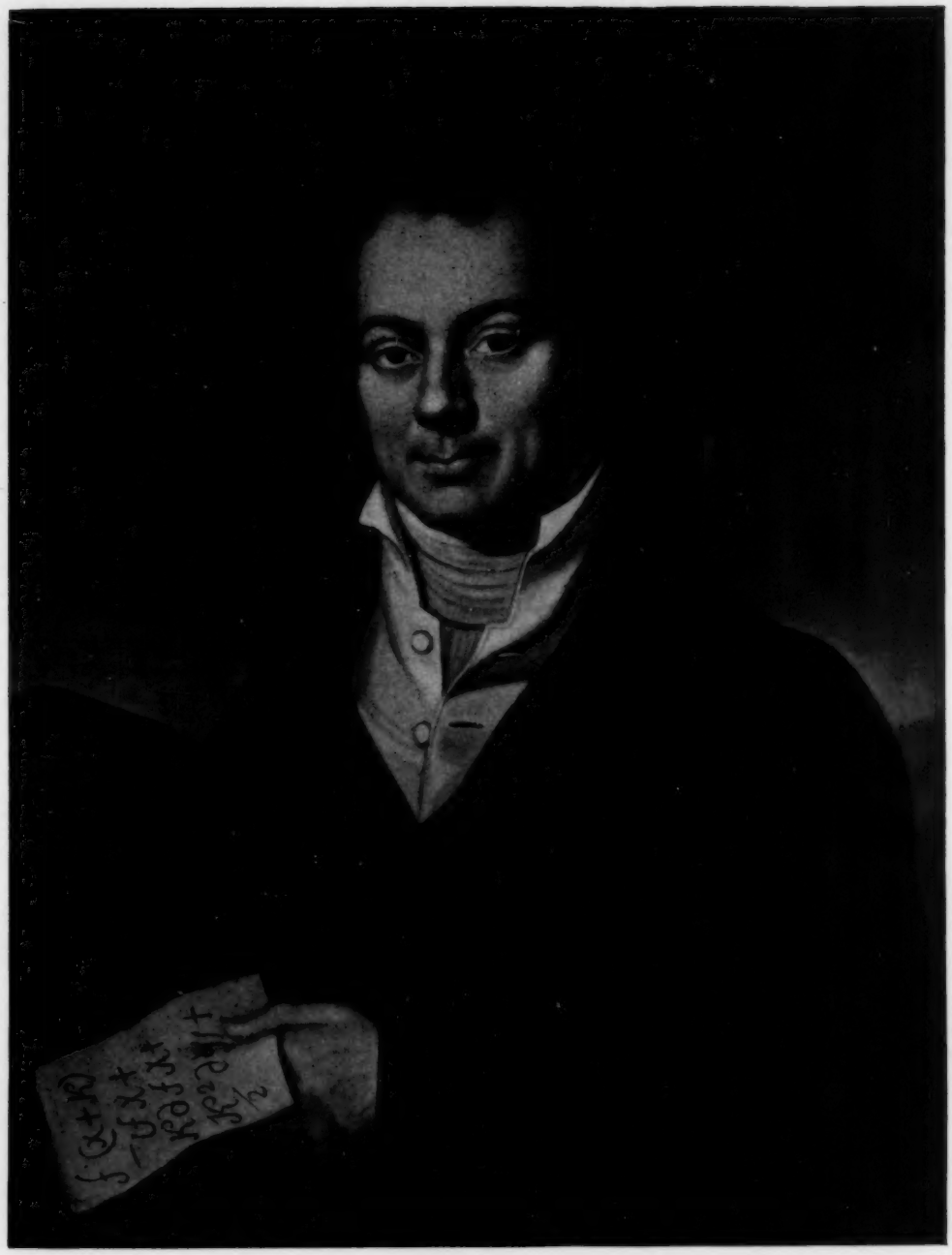
The result actually is the precise state diagram for compact operators. To prove this requires a list of examples to show that each remaining state is possible except as restricted. Since we only need to know which states are not possible for compact operators, we do not give these examples here.

Since Theorem 1 shows that all the bounded operators are compact, the state diagram for compact operators applies. But X' and Y are inseparable and X is reflexive so the only state possible for $A \in [X, Y] = [X, Y]_c$ is $(\mathrm{III}_3, \mathrm{III}_3)$. Thus question 2 is answered.

Bibliography.

[1] *S. Banach*, Theorie des Opérations Linéarès, Warsaw 1932.
[2] *P. Civin* and *B. Yood*, Quasi-Reflexive Spaces, Proc. AMS 8 (1957), 906—911.
[3] *L. W. Cohen* and *N. Dunford*, Transformations on Sequence Spaces, Duke Math. J. 3 (1957).
[4] *M. M. Day*, Normed Linear Spaces, Berlin 1958.
[5] *J. L. Kelley*, General Topology, Van Nostrand 1950.
[6] *R. S. Phillips*, On Linear Transformations, Trans. AMS 48 (1940), 516—541.
[7] *H. R. Pitt*, A Note on Bilinear Forms, J. London Math. Soc. 11 (1936), 174—180.
[8] *A. E. Taylor*, Introduction to Functional Analysis, New York 1958.
[9] *A. E. Taylor* and *C. J. A. Halberg*, General Theorems about a Bounded Linear Operator and its Conjugate, J. Reine Angew. Math. 198 (1957), 93—111.

Eingegangen 20. Mai 1959.



August Leopold Crelle im Alter von etwa 35 Jahren

Ölgemälde um 1815, unbekannter Meister

Bezirksheimatarchiv Berlin-Schöneberg

ge
de
au

vo

un

be

au

zu

lin

In
pr
D

ti
hi
ol
pr
$f($

Verfahren zur Überführung gewisser differentieller Randwertprobleme in lineare Integralgleichungen.

Von *M. Schoch* und *A. Kneschke* in Freiberg/Sachsen.

I. Einleitung.

Schon bevor eine systematische Theorie der linearen Integralgleichungen ausgearbeitet war, hat man auf den Zusammenhang differentieller Randwertprobleme mit den Integralgleichungen hingewiesen und Verfahren zur Überführung linearer Randwertaufgaben in Integralgleichungen ausgebildet.

Bereits 1837 hat J. Liouville [1] ein solches Überführungsverfahren mitgeteilt, das von der für lineare Differentialausdrücke

$$P\{f\} = \sum_{i=0}^{n} a_i(\xi)\, f^{(i)}(\xi)$$

und ihren adjungierten

$$Q\{h\} = \sum_{i=0}^{n} (-1)^i\, [a_i(\xi)\, h(\xi)]^{(i)}$$

bestehenden Relation

$$h(\xi)\, P\{f(\xi)\} - f(\xi)\, Q\{h(\xi)\} = [R\{f(\xi), h(\xi)\}]'$$

ausgeht, wobei

$$R\{f, h\} = \sum_{\nu=1}^{n} Q_\nu\{h\}\, f^{(\nu-1)}$$

$$Q_\nu\{h\} = \sum_{i=0}^{n-\nu} (-1)^i\, [a_{\nu+i}\, h]^{(i)}$$

zu setzen ist.

Mit Hilfe der Greenschen Funktion hat erstmals Burkhardt [2] für gewöhnliche lineare Differentialgleichungen die Überführung in Integralgleichungen durchgeführt.

Bei beiden Verfahren ist es jedoch erforderlich, zur Bestimmung des Kerns der Integralgleichung ein aus dem Ausgangsproblem abzuleitendes spezielles Integrationsproblem zu lösen oder ein Fundamentalsystem von Lösungen einer gewissen homogenen Differentialgleichung zu benutzen.

Auf die Möglichkeit, mittels des Taylorschen Theorems die Überführung differentieller Randwertprobleme in Integralgleichungen durchzuführen, hat G. Kowalewski [3] hingewiesen. Er beschränkt sich dabei auf besonders einfache Randbedingungen und leitet ohne die bisher erforderliche Integration einer Differentialgleichung die dem Randwertproblem n-ter Ordnung äquivalente Integralgleichung für die n-te Ableitung der Lösung $f(x)$ her.

Im Anschluß hieran hat A. Kneschke [4] ein Verfahren entwickelt, das zur Überführung das den Randbedingungen entsprechende Interpolationstheorem verwendet und unmittelbar für $f(x)$ die äquivalente Integralgleichung liefert.

Im folgenden soll dieses Verfahren auf lineare Randwertaufgaben mit sehr allgemeinen Randbedingungen angewendet, die Äquivalenz zwischen dem differentiellen Integrationsproblem und der Integralgleichung nachgewiesen und gezeigt werden, welche Wechselbeziehungen zwischen beiden analytischen Formen der Aufgabenstellung bestehen. Dabei wird ein Zusammenhang mit der Methode der Greenschen Funktion hergestellt.

II. Aufgabenstellung.

Es seien $x_1 < x_2 < \cdots < x_l$ fest vorgegebene Stellen. Jeder dieser Stellen x_μ; $\mu = 1, 2, \ldots, l$ sei eine eigentlich monoton wachsende Teilfolge $\nu_{\mu 1}, \nu_{\mu 2}, \ldots, \nu_{\mu s_\mu}$ der Folge der Zahlen $1, 2, \ldots, n$ zugeordnet. Dabei möge

$$\text{(a)} \qquad \sum_{\mu=1}^{l} s_\mu = n \geqq l$$

und

$$\text{(b)} \qquad \operatorname{Min}\{\nu_{\mu 1}\} = 1, \qquad \operatorname{Max}\{\nu_{\mu s_\mu}\} = r \leqq n$$

gelten. Jede der Zahlen $1, 2, \ldots, r$ soll in wenigstens einer der Folgen $\{\nu_{\mu j}\}$; $j = 1, 2, \ldots, s_\mu$ vorkommen.

Wir legen unseren Betrachtungen die inhomogene lineare Randwertaufgabe

$$\text{(A)} \qquad \begin{aligned} & f^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x)\, f^{(i)}(x) = g(x) \\ & f^{(\nu_{\mu j}-1)}(x_\mu) = f_\mu^{(\nu_{\mu j}-1)}; \qquad \begin{aligned} \mu &= 1, 2, \ldots, l \\ j &= 1, 2, \ldots, s_\mu \end{aligned} \end{aligned}$$

zugrunde, in der die freie Funktion $g(x)$ im Intervall $x_1 \leqq x \leqq x_l$ stetig und die Funktionen $a_i(x)$ ebenda i-mal stetig differenzierbar sein sollen.

Die Gesamtheit der für $f(x)$ und ihre Ableitungen nach (a) und (b) vorgegebenen Werte bezeichnen wir als *dichtes Spektrum* von $f(x)$. Dieses ist also dadurch gekennzeichnet, daß in ihm jede zwischen 0 und $r - 1$ einschließlich liegende Ableitungsstufe mindestens einmal vorkommt. Ist insbesondere für jede Stelle x_μ jede Ableitung von der 0-ten bis zur $(\nu_{\mu s_\mu} - 1)$-ten Ordnung genau einmal vorhanden, dann heiße das Spektrum *gleichmäßig dicht*. Wir nennen ein dichtes Spektrum *ungleichmäßig dicht*, wenn wenigstens in einer der l Ableitungsfolgen eine Lücke auftritt.

Die so formulierte Randwertaufgabe (A) ist in eine Integralgleichung überzuführen.

III. Zugehöriges Interpolationstheorem.

Da das Überführungsverfahren von dem für das dichte Spektrum maßgebenden Interpolationstheorem ausgeht, soll vorerst diese Interpolationsformel aufgestellt werden. Dazu betrachten wir das spezielle Integrationsproblem

$$\text{(B)} \qquad \begin{aligned} & f^{(n)}(x) = g(x) \\ & f^{(\nu_{\mu j}-1)}(x_\mu) = f_\mu^{(\nu_{\mu j}-1)}; \qquad \begin{aligned} \mu &= 1, 2, \ldots, l \\ j &= 1, 2, \ldots, s_\mu \end{aligned} \end{aligned}$$

und nehmen an, $f(x)$ sei eine Lösung von (B). Aus dem Taylorschen Theorem

$$f(x) = \sum_{i=0}^{n-1} f^{(i)}(x_0) \frac{(x-x_0)^i}{i!} + \int_{x_0}^{x} \frac{(x-\xi)^{n-1}}{(n-1)!} f^{(n)}(\xi)\, d\xi$$

mit beliebigem x_0 aus $x_1 \leqq x_0 \leqq x_l$ erhält man durch m-maliges Differenzieren nach x für $0 \leqq m \leqq n-1$

$$f^{(m)}(x) = \sum_{i=0}^{n-1}{}' f^{(i)}(x_0) \frac{(x-x_0)^{i-m}}{(i-m)!} + \int_{x_0}^{x} \frac{(x-\xi)^{n-m-1}}{(n-m-1)!} f^{(n)}(\xi)\, d\xi.$$

Dabei soll hier wie auch stets im folgenden der Strich am Summenzeichen darauf hinweisen, daß für alle Summanden, in denen negative Potenzen von $(x-x_0)$ oder entsprechender Differenzen auftreten, Null zu setzen ist. Wir führen x_μ an Stelle von x und $\nu_{\mu j}-1$ an Stelle von m ein und betrachten x_0, wofür wir x setzen, als Variable. Es gilt dann für $\mu = 1, 2, \ldots, l$; $j = 1, 2, \ldots, s_\mu$

$$f^{(\nu_{\mu j}-1)}(x_\mu) = \sum_{i=0}^{n-1}{}' f^{(i)}(x) \frac{(x_\mu - x)^{i-\nu_{\mu j}+1}}{(i-\nu_{\mu j}+1)!} + \int_{x}^{x_\mu} \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} f^{(n)}(\xi)\, d\xi.$$

Diese Relationen multiplizieren wir mit vorerst noch unbestimmten Funktionen $P_{\mu j}(x)$, die nicht von $f(x)$ und den vorgegebenen Werten $f_\mu^{(\nu_{\mu j}-1)}$ abhängen, und summieren über μ und j, so daß

$$\sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}(x) = \sum_{i=0}^{n-1} f^{(i)}(x) \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' \frac{(x_\mu - x)^{i-\nu_{\mu j}+1}}{(i-\nu_{\mu j}+1)!} P_{\mu j}(x)$$
$$+ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \int_{x}^{x_\mu} \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} f^{(n)}(\xi)\, d\xi$$

folgt. Um aus dieser Gleichung die Ableitungen $f^{(i)}(x)$; $i = 1, 2, \ldots, n-1$ zu beseitigen, stellen wir an die $P_{\mu j}(x)$ die Forderungen

$$(1) \qquad \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' \frac{(x_\mu - x)^{i-\nu_{\mu j}+1}}{(i-\nu_{\mu j}+1)!} P_{\mu j}(x) = \delta_{i0}; \quad i = 0, 1, \ldots, n-1$$

und erhalten unter der Voraussetzung, daß sich solche Funktionen $P_{\mu j}(x)$ finden lassen, das Interpolationstheorem

$$(2) \quad f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x) + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \int_{x_\mu}^{x} \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} f^{(n)}(\xi)\, d\xi.$$

Ist $f(x)$ insbesondere ein Polynom $\varphi(x)$ höchstens $(n-1)$-ten Grades, so geht (2) in

$$\varphi(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \varphi^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}(x)$$

über. Durch q-maliges Differenzieren nach x für $0 \leqq q \leqq n-1$ erhalten wir daraus

$$(3) \qquad \varphi^{(q)}(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \varphi^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}^{(q)}(x).$$

Jetzt wählen wir die Werte $\varphi^{(\nu_{\mu j}-1)}(x_\mu)$ speziell so, daß für feste Zahlen τ und i aus den Folgen $1, 2, \ldots, l$ bzw. $1, 2, \ldots, s_\tau$

$$\varphi^{(\nu_{\mu j}-1)}(x_\mu) = \delta_{\mu\tau}\delta_{ij}; \qquad \begin{array}{l} \mu = 1, 2, \ldots, l, \\ j = 1, 2, \ldots, s_\mu \end{array}$$

gilt. Die Beziehung (3) liefert für $q = \nu_{\tau i} - 1$ und $x = x_\tau$

$$\text{(4a)} \qquad \varphi^{(\nu_{\tau i}-1)}(x_\tau) = 1 = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \varphi^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}^{(\nu_{\tau i}-1)}(x_\tau) = P_{\tau i}^{(\nu_{\tau i}-1)}(x_\tau),$$

für $q = \nu_{\lambda k} - 1$ und $x = x_\lambda$ bei $|\lambda - \tau| + |i - k| > 0$

$$\text{(4b)} \qquad \varphi^{(\nu_{\lambda k}-1)}(x_\lambda) = 0 = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \varphi^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}^{(\nu_{\lambda k}-1)}(x_\lambda) = P_{\tau i}^{(\nu_{\lambda k}-1)}(x_\lambda).$$

Aus (4a) und (4b) folgen die für die $P_{\mu j}(x)$ notwendigen Bedingungen

$$\text{(5)} \qquad P_{\mu j}^{(\nu_{\lambda i}-1)}(x_\lambda) = \delta_{\lambda\mu}\delta_{ij}; \qquad \begin{array}{l} \lambda, \mu = 1, 2, \ldots, l, \\ i = 1, 2, \ldots, s_\lambda, \\ j = 1, 2, \ldots, s_\mu. \end{array}$$

Durch (5) werden an jede der Funktionen $P_{\mu j}(x)$ genau n Forderungen gestellt. Diese lassen sich für Polynome $P_{\mu j}(x)$ von höchstens $(n-1)$-tem Grade gewiß erfüllen. Kommt zu den Bedingungen (5) noch die Forderung des Minimalgrades hinzu, dann sind diese Polynome eindeutig bestimmt [5].

Nun soll gezeigt werden, daß die aus (5) ermittelten Polynome die Relationen (1) erfüllen. Unter unseren Voraussetzungen über die $\nu_{\mu j}$ läßt sich wegen (5) jedes Polynom, dessen Grad höchstens $n-1$ ist, in der Form (3)

$$\psi(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \psi^{(\nu_{\mu j}-1)}(x_\mu)\, P_{\mu j}(x)$$

darstellen, so daß speziell für

$$\psi(x; z) = \frac{(x-z)^p}{p!}; \; p = 0, 1, \ldots, n-1$$

die Relation

$$\frac{(x-z)^p}{p!} = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' \frac{(x_\mu - z)^{p-\nu_{\mu j}+1}}{(p-\nu_{\mu j}+1)!}\, P_{\mu j}(x)$$

gilt. Wir differenzieren diese q-mal nach x für $q = 0, 1, \ldots, p$ und erhalten aus

$$\frac{(x-z)^{p-q}}{(p-q)!} = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' \frac{(x_\mu - z)^{p-\nu_{\mu j}+1}}{(p-\nu_{\mu j}+1)!}\, P_{\mu j}^{(q)}(x)$$

für $z = x$ die Gleichungen

$$\text{(6)} \qquad \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' \frac{(x_\mu - x)^{p-\nu_{\mu j}+1}}{(p-\nu_{\mu j}+1)!}\, P_{\mu j}^{(q)}(x) = \delta_{pq}; \qquad \begin{array}{l} p, q = 0, 1, \ldots, n-1 \\ q \leqq p, \end{array}$$

die für $q = 0$ in (1) übergehen. Damit ist gezeigt, daß die Polynome mit den Eigenschaften (5) diese Gleichungen erfüllen. Darüber hinaus ist nun auch die Gültigkeit der Beziehung (2) nachgewiesen, die das Interpolationstheorem für ein ungleichmäßig dichtes Funktionsspektrum darstellt. M. Schoch [5] hat dieses allgemeine Theorem auf direktem Wege hergeleitet.

Man überzeugt sich leicht, daß die durch

$$(7)\qquad f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x) + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \int_{x_\mu}^{x} \frac{(x_\mu - \xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!}\, g(\xi)\, d\xi$$

gegebene Funktion die Differentialgleichung von (B) erfüllt. Bei n-maligem Differenzieren erhalten wir aus

$$f^{(n)}(x) = \frac{d^n}{dx^n} \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \int_{x_\mu}^{x} \frac{(x_\mu - \xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!}\, g(\xi)\, d\xi$$

unter Anwendung von (6)

$$f^{(n)}(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}^{(n-1)}(x) \frac{(x_\mu - x)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!}\, g(x) = g(x).$$

Daß die Randbedingungen erfüllt sind, folgt unmittelbar aus der Interpolationsformel (2). Die Funktion (7) stellt die einzige Lösung der Integrationsaufgabe (B) dar. Sie dient zur Überführung des allgemeinen Randwertproblems (A) in eine Integralgleichung.

Im Falle eines gleichmäßig dichten Spektrums

$$\nu_{\mu j} = j;\ \mu = 1, 2, \ldots, l;\ j = 1, 2, \ldots, s_\mu$$

ergibt sich die Lösung

$$f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(j-1)} P_{\mu j}(x) + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \int_{x_\mu}^{x} \frac{(x_\mu - \xi)^{n-j}}{(n-j)!}\, g(\xi)\, d\xi,$$

die für $g(\xi) = f^{(n)}(\xi)$ die zugehörige Interpolationsformel darstellt. Dabei erfüllen die Interpolationspolynome die Bedingungen

$$P_{\mu j}^{(i-1)}(x_\lambda) = \delta_{\lambda\mu}\delta_{ij}; \qquad \begin{aligned} &\lambda, \mu = 1, 2, \ldots, l \\ &i = 1, 2, \ldots, s_\lambda \\ &j = 1, 2, \ldots, s_\mu. \end{aligned}$$

Die Lösung (7) der speziellen Randwertaufgabe läßt sich in der Form

$$(7\text{a})\qquad f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x) + \int_{x_1}^{x_l} H(x, \xi)\, g(\xi)\, d\xi$$

darstellen, wobei

$$(8)\qquad H(x, \xi) = \begin{cases} \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x)\, [-1 - \operatorname{sgn}(x_\mu - \xi)]\, \dfrac{(x_\mu - \xi)^{n-\nu_{\mu j}}}{2(n-\nu_{\mu j})!}; & x \leqq \xi \\ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x)\, [1 - \operatorname{sgn}(x_\mu - \xi)]\, \dfrac{(x_\mu - \xi)^{n-\nu_{\mu j}}}{2(n-\nu_{\mu j})!}; & x > \xi \end{cases}$$

ist.

Es soll nun gezeigt werden, daß $H(x, \xi)$ die Greensche Funktion der Randwertaufgabe

$$(\text{B}^*)\qquad \begin{aligned} f_1^{(n)}(x) &= g(x) \\ f_1^{(\nu_{\mu j}-1)}(x_\mu) &= 0 \end{aligned}$$

ist. Diese Aufgabe ist eindeutig lösbar, da das entsprechende vollhomogene Problem nur die triviale Lösung besitzt. Die die Greensche Funktion kennzeichnenden Eigenschaften lassen sich leicht nachweisen.

$H(x, \xi)$ besitzt in jedem der Dreiecksbereiche $x_1 \leqq x < \xi \leqq x_l$ und $x_1 \leqq \xi < x \leqq x_l$ die partiellen Ableitungen

$$\frac{\partial^\nu}{\partial x^\nu} H(x, \xi); \quad \nu = 0, 1, \ldots, n.$$

Für $\nu = 0, 1, \ldots, n-2$ existieren diese Ableitungen sogar im ganzen Bereich $x_1 \leqq x, \xi \leqq x_l$ und sind dort stetig.

Da die $P_{\mu j}(x)$ sämtlich höchstens den Grad $(n-1)$ haben, erfüllt $H(x, \xi)$ in jedem der Dreiecksbereiche als Funktion von x die Differentialgleichung

$$f_1^{(n)}(x) = 0.$$

Wegen (6) ist $H(x, \xi)$ im ganzen Bereich $x_1 \leqq x, \xi \leqq x_l$ stetig.

Für $x_1 < \xi < x_l$ gilt ebenfalls nach (6)

$$\frac{\partial^{n-1}}{\partial x^{n-1}} H(x, \xi) \Big|_{x=\xi-0}^{x=\xi+0} = 1.$$

Für jedes feste ξ aus (x_1, x_l) ist

$$\frac{\partial^{\nu_{\lambda i}-1}}{\partial x^{\nu_{\lambda i}-1}} H(x, \xi) \Big|_{x=x_\lambda} = 0.$$

Die Lösung der Aufgabe (B*) lautet demnach

$$f_1(x) = \int_{x_1}^{x_l} H(x, \xi)\, g(\xi)\, d\xi.$$

Aus ihr erhält man durch Addition des Interpolationspolynoms

$$Q(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x)$$

die Lösung (7a) des Problems (B).

IV. Überführung des Problems (A) in eine Integralgleichung.

Um nun die dem Randwertproblem (A) äquivalente Integralgleichung zu erhalten, unterwirft man die Differentialgleichung

$$f^{(n)}(\xi) + \sum_{i=0}^{n-1} a_i(\xi)\, f^{(i)}(\xi) = g(\xi)$$

der Integraloperation

$$\sum_{\mu=1}^{l} \int_{x_\mu}^{x} \cdots R_\mu(x, \xi)\, d\xi,$$

in der

$$R_\mu(x, \xi) = \sum_{j=1}^{s_\mu} P_{\mu j}(x) \frac{(x_\mu - \xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} \tag{9}$$

mit den in (7) auftretenden Polynomen $P_{\mu j}(x)$ gegeben ist. Mit den Abkürzungen

$$J_n(x) = \sum_{\mu=1}^{l} \int_{x_\mu}^{x} f^{(n)}(\xi)\, R_\mu(x, \xi)\, d\xi$$

$$J_i(x) = \sum_{\mu=1}^{l} \int_{x_\mu}^{x} a_i(\xi)\, f^{(i)}(\xi)\, R_\mu(x, \xi)\, d\xi;\ i = 0, 1, \ldots, n-1$$

erhalten wir dadurch

$$J_n(x) + \sum_{i=0}^{n-1} J_i(x) = \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi. \tag{10}$$

Aus dem Interpolationsproblem (2) folgt

$$J_n(x) = f(x) - \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x). \tag{11a}$$

Durch partielle Integration ergibt sich

$$J_i(x) = \sum_{\mu=1}^{l} \sum_{\lambda=0}^{i-1} (-1)^\lambda f^{(i-\lambda-1)}(\xi)\, [a_i(\xi)\, R_\mu(x, \xi)]^{(\lambda)} \Big|_{\xi=x_\mu}^{\xi=x}$$
$$+ (-1)^i \sum_{\mu=1}^{l} \int_{x_\mu}^{x} f(\xi)\, [a_i(\xi)\, R_\mu(x, \xi)]^{(i)}\, d\xi.$$

Dabei ist die Differentiation in den eckigen Klammern nach der doppelt vorkommenden Variablen auszuführen. Wir wenden auf den integralfreien Summanden die Leibnizsche Produktregel an, berücksichtigen die sich aus (6) ergebenden Beziehungen

$$\sum_{\mu=1}^{l} \left\{ \frac{\partial^\lambda}{\partial \xi^\lambda} R_\mu(x, \xi) \right\}_{\xi=x} = (-1)^\lambda \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu}{}' P_{\mu j}(x) \frac{(x_\mu - x)^{n-\nu_{\mu j}-\lambda}}{(n-\nu_{\mu j}-\lambda)!} = (-1)^\lambda \delta_{\lambda, n-1} \tag{c}$$

$$\frac{\partial^\lambda}{\partial \xi^\lambda} R_\mu(x, \xi) \Big|_{\xi=x_\mu} = (-1)^\lambda \sum_{j=1}^{s_\mu} P_{\mu j}(x)\, \delta_{\lambda, n-\nu_{\mu j}} \tag{d}$$

und erhalten, da $\lambda \leqq i-1 \leqq n-2$ ist, für $i = 0, 1, \ldots, n-1$

$$J_i(x) = -\sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} \sum_{\lambda=n-\nu_{\mu j}}^{i-1} (-1)^{\lambda+n-\nu_{\mu j}} \binom{\lambda}{n-\nu_{\mu j}} f^{(i-\lambda-1)}(x_\mu)\, a_i^{(\lambda+\nu_{\mu j}-n)}(x_\mu)\, P_{\mu j}(x) \tag{11b}$$
$$+ (-1)^i \sum_{\mu=1}^{l} \int_{x_\mu}^{x} f(\xi)\, [a_i(\xi)\, R_\mu(x, \xi)]^{(i)}\, d\xi.$$

Mit (11a) und (11b) geht (10), wenn wir die Indizes gemäß $\varrho = \lambda - n + \nu_{\mu j}$ und $\sigma = i - n + \nu_{\mu j} - 1$ verschieben und

$$\binom{\varrho + n - \nu_{\mu j}}{n - \nu_{\mu j}} = \binom{\varrho + n - \nu_{\mu j}}{\varrho}$$

beachten, in die lineare Integralgleichung

$$f(x) + \sum_{\mu=1}^{l} \int_{x_\mu}^{x} f(\xi) \sum_{i=0}^{n-1} (-1)^i [a_i(\xi)\, R_\mu(x, \xi)]^{(i)}\, d\xi$$
$$= \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x)$$
$$+ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \sum_{\sigma=0}^{\nu_{\mu j}-2} \sum_{\varrho=0}^{\sigma} (-1)^\varrho \binom{\varrho + n - \nu_{\mu j}}{\varrho} f^{(\sigma-\varrho)}(x_\mu)\, a_{\sigma+n-\nu_{\mu j}+1}^{(\varrho)}(x_\mu)$$

über. Dabei soll hier und auch stets im folgenden eine Summe, deren obere Summationsgrenze kleiner als die untere ist, identisch Null sein. Die im letzten Summanden auftretende innere Doppelsumme S_0 wollen wir in die Form

$$S_0 = \sum_{\nu=0}^{\nu_{\mu j}-2} f^{(\nu)}(x_\mu)\, A_\nu$$

bringen. Mit $\varrho = \sigma - \nu$ folgt für festes ν aus $0 \leqq \nu \leqq \nu_{\mu j} - 2$

$$f^{(\nu)}(x_\mu) \sum_{\sigma=\nu}^{\nu_{\mu j}-2} (-1)^{\sigma-\nu} \binom{\sigma - \nu + n - \nu_{\mu j}}{\sigma - \nu} a^{(\sigma-\nu)}_{\sigma+n-\nu_{\mu j}+1}(x_\mu)$$

oder

$$f^{(\nu)}(x_\mu) \sum_{\tau=0}^{\nu_{\mu j}-\nu-2} (-1)^{\tau} \binom{\tau + n - \nu_{\mu j}}{\tau} a^{(\tau)}_{n-\nu_{\mu j}+\tau+\nu+1}(x_\mu) = f^{(\nu)}(x_\mu)\, A_\nu .$$

Es ist also nach Ersetzen von ν durch σ und τ durch ϱ und Summation über σ

$$S_0 = \sum_{\sigma=0}^{\nu_{\mu j}-2} f^{(\sigma)}(x_\mu) \sum_{\varrho=0}^{\nu_{\mu j}-\sigma-2} (-1)^{\varrho} \binom{\varrho + n - \nu_{\mu j}}{\varrho} a^{(\varrho)}_{n-\nu_{\mu j}+\varrho+\sigma+1}(x_\mu).$$

Dies tragen wir in den letzten Summanden S_1 ein und trennen die vorgegebenen Ableitungswerte $f^{(\nu_{\mu j}-1)}(x_\mu) = f_\mu^{(\nu_{\mu j}-1)}$ von den nicht vorgegebenen. Dann ergibt sich

$$\begin{aligned}(12)\quad S_1 = {} & \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ \sum_{k=1}^{j-1} f_\mu^{(\nu_{\mu k}-1)} \sum_{\varrho=0}^{\nu_{\mu j}-\nu_{\mu k}-1} (-1)^{\varrho} \binom{\varrho + n - \nu_{\mu j}}{\varrho} a^{(\varrho)}_{n-\nu_{\mu j}+\nu_{\mu k}+\varrho}(x_\mu) \right. \\ & \left. + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ \sum_{\substack{\sigma=0 \\ \sigma \neq \nu_{\mu s}-1}}^{\nu_{\mu j}-2} f^{(\sigma)}(x_\mu) \sum_{\varrho=0}^{\nu_{\mu j}-\sigma-2} (-1)^{\varrho} \binom{\varrho + n - \nu_{\mu j}}{\varrho} a^{(\varrho)}_{n-\nu_{\mu j}+\sigma+\varrho+1}(x_\mu) \right\}.\end{aligned}$$

Dabei sind bei der Summation über σ alle die Indizes auszulassen, für die $\sigma = \nu_{\mu s} - 1$; $s = 1, 2, \ldots, j-1$ gilt. Damit erhält die Integralgleichung die Form

$$(13)\qquad f(x) + \sum_{\mu=1}^{l} \int_{x_\mu}^{x} K_\mu(x, \xi)\, f(\xi)\, d\xi = G(x)$$

mit

$$K_\mu(x, \xi) = \sum_{i=0}^{n-1} (-1)^i [a_i(\xi)\, R_\mu(x, \xi)]^{(i)}$$

und der freien Funktion

$$\begin{aligned}(13\text{a})\quad G(x) = {} & \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi \\ & + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ f_\mu^{(\nu_{\mu j}-1)} + \sum_{k=1}^{j-1} f_\mu^{(\nu_{\mu k}-1)} \sum_{\varrho=0}^{\nu_{\mu j}-\nu_{\mu k}-1} (-1)^{\varrho} \binom{\varrho + n - \nu_{\mu j}}{\varrho} \right. \\ & \qquad \left. a^{(\varrho)}_{n-\nu_{\mu j}+\nu_{\mu k}+\varrho}(x_\mu) \right\} \\ & + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ \sum_{\substack{\sigma=0 \\ \sigma \neq \nu_{\mu s}-1}}^{\nu_{\mu j}-2} f^{(\sigma)}(x_\mu) \sum_{\varrho=0}^{\nu_{\mu j}-\sigma-2} (-1)^{\varrho} \binom{\varrho + n - \nu_{\mu j}}{\varrho} \right. \\ & \qquad \left. a^{(\varrho)}_{n-\nu_{\mu j}+\sigma+\varrho+1}(x_\mu) \right\}.\end{aligned}$$

Die Gleichung (13) ist im allgemeinen nicht eine Integralgleichung im gewöhnlichen Sinne, da im letzten Summanden der freien Funktion $G(x)$ Ableitungen der gesuchten Funktion $f(x)$ an festen Stellen auftreten. Wir haben durch die bisherigen Betrachtungen festgestellt, daß jede Lösung $f(x)$ des differentiellen Randwertproblems (A) die Integralgleichung (13) erfüllt.

Bei einem gleichmäßig dichten Spektrum ($\nu_{\mu j} = j$) treten in der freien Funktion keine Ableitungen der gesuchten Funktion an festen Stellen auf. In diesem Sonderfall gehört zur Randwertaufgabe

$$f^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x)\, f^{(i)}(x) = g(x)$$

$$\text{(A*)} \qquad f^{(j-1)}(x_\mu) = f_\mu^{(j-1)}; \qquad \begin{array}{l} \mu = 1, 2, \ldots, l \\ j = 1, 2, \ldots, s_\mu, \end{array}$$

wie bereits A. Kneschke [4] gezeigt hat, eine Integralgleichung der Form (13) mit der einfacheren freien Funktion

$$G(x) = \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi$$
$$+ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ f_\mu^{(j-1)} + \sum_{k=1}^{j-1} f_\mu^{(k-1)} \sum_{\varrho=0}^{j-k-1} (-1)^\varrho \binom{\varrho + n - j}{\varrho} a_{n-j+k+\varrho}^{(\varrho)}(x_\mu) \right\}.$$

Man kann zeigen, daß umgekehrt jede Lösung der Integralgleichung (13) die Randwertaufgabe (A) befriedigt. Dazu integriert man das Kernintegral von (13) partiell, wandelt das integralfreie Glied unter Beachtung von (c) und (d) in die Form (12) um und erhält

$$f(x) + \sum_{\mu=1}^{l} \sum_{i=0}^{n-1} \int_{x_\mu}^{x} f^{(i)}(\xi)\, a_i(\xi)\, R_\mu(x, \xi)\, d\xi$$
$$= \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \left\{ f_\mu^{(\nu_{\mu j}-1)} + \sum_{k=1}^{j-1} \left(f_\mu^{(\nu_{\mu k}-1)} - f^{(\nu_{\mu k}-1)}(x_\mu) \right) \right.$$
$$\left. \cdot \sum_{\varrho=0}^{\nu_{\mu j}-\nu_{\mu k}-1} (-1)^\varrho \binom{\varrho + n - \nu_{\mu j}}{\varrho} a_{n-\nu_{\mu j}+\nu_{\mu k}+\varrho}^{(\varrho)}(x_\mu) \right\}.$$

Dabei sind mit $f_\mu^{(\nu_{\mu k}-1)}$ die in (A) auftretenden Konstanten bezeichnet, während $f^{(\nu_{\mu k}-1)}(x_\mu)$ die $(\nu_{\mu k} - 1)$-te Ableitung der Lösung $f(x)$ der Integralgleichung an der Stelle x_μ bedeutet. Nach m-maligem Differenzieren nach x ergibt sich, weil man mit Rücksicht auf

$$\sum_{\mu=1}^{l} \left[\frac{\partial^\lambda}{\partial x^\lambda} R_\mu(x, \xi) \right]_{\xi = x} = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}^{(\lambda)}(x) \frac{(x_\mu - x)^{n-\nu_{\mu j}}}{(n - \nu_{\mu j})!} = \delta_{\lambda, n-1}$$

für $0 \leqq m \leqq n - 1$ die Beziehungen

$$\text{(e)} \qquad \begin{aligned} \sum_{\mu=1}^{l} \frac{d^m}{dx^m} \int_{x_\mu}^{x} f^{(i)}(\xi)\, a_i(\xi)\, R_\mu(x, \xi)\, d\xi &= \sum_{\mu=1}^{l} \int_{x_\mu}^{x} f^{(i)}(\xi)\, a_i(\xi) \frac{\partial^m}{\partial x^m} R_\mu(x, \xi)\, d\xi \\ \sum_{\mu=1}^{l} \frac{d^m}{dx^m} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi &= \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi) \frac{\partial^m}{\partial x^m} R_\mu(x, \xi)\, d\xi \end{aligned}$$

bestätigt,

$$(14)\quad f^{(m)}(x) + \sum_{\mu=1}^{l} \sum_{i=0}^{n-1} \int_{x_\mu}^{x} f^{(i)}(\xi)\, a_i(\xi) \frac{\partial^m}{\partial x^m} R_\mu(x,\xi)\, d\xi$$

$$= \sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi) \frac{\partial^m}{\partial x^m} R_\mu(x,\xi)\, d\xi + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}^{(m)}(x) \Big\{ f^{(\nu_{\mu j}-1)} + \sum_{k=1}^{j-1} (f_\mu^{(\nu_{\mu k}-1)} - f^{(\nu_{\mu k}-1)}(x_\mu))$$

$$\cdot \sum_{\varrho=0}^{\nu_{\mu j}-\nu_{\mu k}-1} (-1)^\varrho \binom{\varrho+n-\nu_{\mu j}}{\varrho} a_{n-\nu_{\mu j}+\nu_{\mu k}+\varrho}^{(\varrho)}(x_\mu) \Big\}.$$

Setzt man nun $m = n-1$, so folgt durch nochmaliges Differenzieren wegen

$$P_{\mu j}^{(n)}(x) \equiv 0, \quad \frac{\partial^n}{\partial x^n} R_\mu(x,\xi) \equiv 0, \quad \sum_{\mu=1}^{l} \left[\frac{\partial^{n-1}}{\partial x^{n-1}} R_\mu(x,\xi) \right]_{\xi=x} = 1,$$

daß jede Lösung $f(x)$ der Integralgleichung die Differentialgleichung

$$f^{(n)}(x) + \sum_{i=0}^{n-1} a_i(x)\, f^{(i)}(x) = g(x)$$

erfüllt.

Für die festen Stellen x_λ; $\lambda = 1, 2, \ldots, l$ und für $m = \nu_{\lambda q} - 1$; $q = 1, 2, \ldots, s_\lambda$ folgt wegen (5) und

$$\left[\frac{\partial^{\nu_{\lambda q}-1}}{\partial x^{\nu_{\lambda q}-1}} R_\mu(x,\xi) \right]_{x=x_\lambda} = \sum_{j=1}^{s_\mu} P_{\mu j}^{(\nu_{\lambda q}-1)}(x_\lambda) \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} = 0; \quad \mu \neq \lambda$$

aus (14)

$$f^{(\nu_{\lambda q}-1)}(x_\lambda)$$

$$= f_\lambda^{(\nu_{\lambda q}-1)} + \sum_{k=1}^{q-1} (f_\lambda^{(\nu_{\lambda k}-1)} - f^{(\nu_{\lambda k}-1)}(x_\lambda)) \sum_{\varrho=0}^{\nu_{\lambda q}-\nu_{\lambda k}-1} (-1)^\varrho \binom{\varrho+n-\nu_{\lambda q}}{\varrho} a_{n-\nu_{\lambda q}+\nu_{\lambda k}+\varrho}^{(\varrho)}(x_\lambda).$$

Die Summe verschwindet für $q = 1$ wegen $q - 1 = 0 < 1$, so daß

$$f^{(\nu_{\lambda 1}-1)}(x_\lambda) = f_\lambda^{(\nu_{\lambda 1}-1)}$$

gilt. Unter der Annahme, daß

$$f^{(\nu_{\lambda s}-1)}(x_\lambda) = f_\lambda^{(\nu_{\lambda s}-1)}; \quad s = 1, 2, \ldots, q-1$$

ist, folgt

$$f^{(\nu_{\lambda q}-1)}(x_\lambda) = f_\lambda^{(\nu_{\lambda q}-1)}; \quad \lambda = 1, 2, \ldots, l; \; q = 1, 2, \ldots, s_\lambda.$$

Jede Lösung der Integralgleichung erfüllt demnach auch die Nebenbedingungen von (A). Damit ist die Äquivalenz des Randwertproblems (A) und der Integralgleichung (13) bewiesen.

Mittels der für $\xi = x$ unstetigen Kernfunktion

$$(15)\qquad K(x,\xi) = \begin{cases} \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x)\, \overset{1}{\psi}_{\mu j}(\xi); & \xi < x \\ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x)\, \overset{2}{\psi}_{\mu j}(\xi); & \xi > x, \end{cases}$$

worin

$$\overset{1}{\psi}_{\mu j}(\xi) = \frac{1}{2} [1 - \operatorname{sgn}(x_\mu - \xi)] \sum_{i=0}^{n-1} (-1)^i \left[a_i(\xi) \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} \right]^{(i)}$$

$$\overset{2}{\psi}_{\mu j}(\xi) = \frac{1}{2} [-1 - \operatorname{sgn}(x_\mu - \xi)] \sum_{i=0}^{n-1} (-1)^i \left[a_i(\xi) \frac{(x_\mu-\xi)^{n-\nu_{\mu j}}}{(n-\nu_{\mu j})!} \right]^{(i)}$$

zu setzen ist, geht die Integralgleichung (13) in die Fredholmsche Form

$$f(x) + \int_{x_1}^{x_l} K(x, \xi)\, f(\xi)\, d\xi = G(x) \tag{16}$$

über. Dabei ist jedoch zu beachten, daß die freie Funktion $G(x)$ von Ableitungen der gesuchten Funktion an festen Stellen abhängt.

Auch für die Aufgabe (A) kann man die zugehörige Integralgleichung mittels der Greenschen Funktion $H(x, \xi)$ für den Differentialausdruck

$$L\{f\} = f^{(n)}(x)$$

erhalten. Falls eine Lösung der Aufgabe (A) existiert, hat sie nämlich analog zu (7a) die Gestalt

$$f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x) + \int_{x_1}^{x_l} H(x, \xi) \left[g(\xi) - \sum_{i=0}^{n-1} a_i(\xi)\, f^{(i)}(\xi)\right] d\xi$$

oder

$$f(x) + \int_{x_1}^{x_l} H(x, \xi) \sum_{i=0}^{n-1} a_i(\xi)\, f^{(i)}(\xi)\, d\xi = \int_{x_1}^{x_l} H(x, \xi)\, g(\xi)\, d\xi - \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x).$$

Durch Aufspalten der Integrale finden wir

$$f(x) + \sum_{\mu=1}^{l} \int_{x_\mu}^{x} R_\mu(x, \xi) \sum_{i=0}^{n-1} a_i(\xi)\, f^{(i)}(\xi)\, d\xi = \sum_{\mu=1}^{l} \int_{x_\mu}^{x} R_\mu(x, \xi)\, g(\xi)\, d\xi + \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x),$$

und diese Gleichung läßt sich, wie bereits durchgeführt wurde, in die Integralgleichung (13) bzw. (16) überführen.

Wegen der Äquivalenz von (A) und (16) ist es möglich, aus der Integralgleichung die differentielle Randwertaufgabe zurückzugewinnen.

Aus

$$\frac{\partial^\nu}{\partial x^\nu} K(x, \xi) \Big|_{\xi = x+0}^{\xi = x-0} = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}^{(\nu)}(x) \left[\overset{1}{\psi}_{\mu j}(x) - \overset{2}{\psi}_{\mu j}(x)\right]; \quad \nu = 0, 1, \ldots, n-1$$

erhält man wegen

$$\overset{1}{\psi}_{\mu j}(x) = \frac{1}{2}\,[1 - \operatorname{sgn}(x_\mu - x)] \sum_{i=0}^{n-1} \sum_{q=0}^{i}{}' (-1)^{i+q} \binom{i}{q} a_i^{(i-q)}(x)\, \frac{(x_\mu - x)^{n-\nu_{\mu j}-q}}{(n-\nu_{\mu j}-q)!}$$

$$\overset{2}{\psi}_{\mu j}(x) = \frac{1}{2}\,[-1 - \operatorname{sgn}(x_\mu - x)] \sum_{i=0}^{n-1} \sum_{q=0}^{i}{}' (-1)^{i+q} \binom{i}{q} a_i^{(i-q)}(x)\, \frac{(x_\mu - x)^{n-\nu_{\mu j}-q}}{(n-\nu_{\mu j}-q)!}$$

bei Beachtung von (6)

$$\frac{\partial^\nu}{\partial x^\nu} K(x, \xi) \Big|_{\xi = x+0}^{\xi = x-0} = \sum_{i=n-\nu-1}^{n-1} (-1)^{i+n-\nu-1} \binom{i}{n-\nu-1} a_i^{(i-n+\nu+1)}(x)$$

oder

$$\frac{\partial^\nu}{\partial x^\nu} K(x, \xi) \Big|_{\xi = x+0}^{\xi = x-0} = \sum_{\lambda=0}^{\nu} (-1)^\lambda \binom{\lambda+n-\nu-1}{n-\nu-1} a_{\lambda+n-\nu-1}^{(\lambda)}(x); \quad \nu = 0, 1, \ldots, n-1. \tag{17a}$$

Aus dem Unstetigkeitsverhalten des Kernes für $\xi = x$ sind hiernach die Koeffizienten $a_i(x)$ rekursiv zu bestimmen.

Die freie Funktion $g(x)$ der Differentialgleichung erhält man bei Beachtung von (e) aus

$$G^{(n)}(x) = \sum_{\mu=1}^{l} \frac{d^n}{dx^n} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi = \sum_{\mu=1}^{l} \frac{d}{dx} \int_{x_\mu}^{x} g(\xi)\, \frac{\partial^{n-1}}{\partial x^{n-1}} R_\mu(x, \xi)\, d\xi$$

wegen

$$\sum_{\mu=1}^{l} \left[\frac{\partial^{n-1}}{\partial x^{n-1}} R_\mu(x, \xi)\right]_{\xi=x} = 1, \quad \frac{\partial^n}{\partial x^n} R_\mu(x, \xi) = 0$$

zu

$$g(x) = G^{(n)}(x). \tag{17b}$$

Schließlich kann man durch Differenzieren von $G(x)$ die Nebenbedingungen von (A) finden. Setzt man in dem Ausdruck von $G^{(m)}(x)$; $0 \leqq m \leqq n-1$ für x den festen Wert x_λ; $\lambda = 1, 2, \ldots, l$ und $m = \nu_{\lambda i} - 1$; $i = 1, 2, \ldots, s_\lambda$, dann ergibt sich bei Beachtung von (5)

$$\begin{aligned} f_\lambda^{(\nu_{\lambda i}-1)} = G^{(\nu_{\lambda i}-1)}(x_\lambda) &- \sum_{k=1}^{i-1} f_\lambda^{(\nu_{\lambda k}-1)} \sum_{\varrho=0}^{\nu_{\lambda i}-\nu_{\lambda k}-1} (-1)^\varrho \binom{\varrho + n - \nu_{\lambda i}}{\varrho} a_{n-\nu_{\lambda i}+\nu_{\lambda k}+\varrho}^{(\varrho)}(x_\lambda) \\ &- \sum_{\substack{\sigma=0 \\ \sigma \neq \nu_{\mu s}-1}}^{\nu_{\lambda i}-2} f^{(\sigma)}(x_\lambda) \sum_{\varrho=0}^{\nu_{\mu i}-\sigma-2} (-1)^\varrho \binom{\varrho + n - \nu_{\lambda i}}{\varrho} a_{n-\nu_{\lambda i}+\sigma+\varrho+1}^{(\varrho)}(x_\lambda). \end{aligned} \tag{17c}$$

Daraus sind die Werte $f_\lambda^{(\nu_{\lambda i}-1)}$ schrittweise zu berechnen.

Die Integralgleichung (13) erhält mit der Abkürzung

$$\begin{aligned} Q(x) = &\sum_{\mu=1}^{l} \int_{x_\mu}^{x} g(\xi)\, R_\mu(x, \xi)\, d\xi \\ &+ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \sum_{k=1}^{j-1} f_\mu^{(\nu_{\mu j}-1)} \sum_{\varrho=0}^{\nu_{\mu j}-\nu_{\mu k}-1} (-1)^\varrho \binom{\varrho + n - \nu_{\mu j}}{\varrho} a_{n-\nu_{\mu j}+\nu_{\mu k}+\varrho}^{(\varrho)}(x_\mu) \\ &+ \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} P_{\mu j}(x) \sum_{\substack{\sigma=0 \\ \sigma \neq \nu_{\mu s}-1}}^{\nu_{\mu j}-2} f^{(\sigma)}(x_\mu) \sum_{\varrho=0}^{\nu_{\mu j}-\sigma-2} (-1)^\varrho \binom{\varrho + n - \nu_{\mu j}}{\varrho} a_{n-\nu_{\mu j}+\sigma+\varrho+1}^{(\varrho)}(x_\mu) \end{aligned}$$

die Gestalt

$$f(x) + \sum_{\mu=1}^{l} \int_{x_\mu}^{x} K_\mu(x, \xi)\, f(\xi)\, d\xi = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} P_{\mu j}(x) + Q(x). \tag{18}$$

Sind $\psi_{\lambda i}(x)$ bzw. $\eta(x)$ Lösungen der Integralgleichungen

$$\psi_{\lambda i}(x) + \sum_{\mu=1} \int_{x_\mu}^{x} K_\mu(x, \xi)\, \psi_{\lambda i}(\xi)\, d\xi = P_{\lambda i}(x); \quad \begin{matrix} \lambda = 1, 2, \ldots, l \\ i = 1, 2, \ldots, s_\lambda \end{matrix} \tag{19}$$

bzw.

$$\eta(x) + \sum_{\mu=1}^{l} \int_{x_\mu}^{x} K_\mu(x, \xi)\, \eta(\xi)\, d\xi = Q(x), \tag{20}$$

dann ist

$$f(x) = \sum_{\mu=1}^{l} \sum_{j=1}^{s_\mu} f_\mu^{(\nu_{\mu j}-1)} \psi_{\mu j}(x) + \eta(x) \tag{21}$$

Lösung von (18) und damit von (13). Da

$$\psi_{\mu j}^{(\nu_{\lambda i}-1)}(x_\lambda) = \delta_{\lambda\mu}\delta_{ij}, \qquad \begin{array}{l} \lambda, \mu = 1, 2, \ldots, l \\ i = 1, 2, \ldots, s_\lambda \\ j = 1, 2, \ldots, s_\mu \end{array}$$

und

$$\eta^{(\nu_{\lambda i}-1)}(x_\lambda) = 0$$

gilt, hat die Lösung $f(x)$ des gestellten Randwertproblems den Charakter einer Interpolationsformel. Die Interpolationsfunktionen $\psi_{\lambda i}(x)$ bestimmen sich nach (19) aus den Newtonschen Interpolationspolynomen $P_{\lambda i}(x)$ der Formel (2) und den Koeffizienten $a_i(x)$ der Differentialgleichung von (A).

Literaturverzeichnis.

[1] *J. Liouville*, Sur le développement des fonctions ou parties de fonctions en séries dont les divers termes sont assujettis à satisfaire à une même équation différentielle du second ordre contenant un paramètre variable. II. Journal de math. (1) **2** (1837), 16—35.

[2] *H. Burkhardt*, Sur les fonctions de Green relatives à un domaine d'une dimension. Bulletin Soc. math. France **22** (1894), 71—75.

[3] *G. Kowalewski*, Überführung linearer Differentialgleichungen in Integralgleichungen. Leipz. Akademieberichte, Math.-phys. Kl. **80** (1928).

[4] *A. Kneschke*, Wechselbeziehungen zwischen Differentialgleichungen und Integralgleichungen (1. Mitteilung), Dtsch. Math. **5** (1941) und (2. Mitteilung), Dtsch. Math. **6** (1942).

[5] *M. Schoch*, Überführung gewisser differentieller Randwertaufgaben in lineare Integralgleichungen. Dissertation. Freiberg 1958.

Eingegangen 9. Mai 1959.

Einige Bemerkungen über inhomogene Linearformen.

Von *G. J. Rieger* in College Park (Maryland).

Es seien $L_1, \ldots, L_n$ homogene Linearformen in den Veränderlichen $x_1, \ldots, x_n$ mit der Determinante $D \neq 0$. Wir nehmen an, daß $L_1, \ldots, L_r$ reelle, $L_{r+1}, \ldots, L_{r+s}$ komplexe (d. h. mindestens einen nichtreellen) Koeffizienten habe und daß $L_{r+s+j} = \overline{L_{r+j}}$ (konjugiert komplex) $(j = 1, \ldots, s)$ ist; $r + 2s = n$. Es bezeichne

$$M_H = \min \mid L_1 \cdots L_n \mid$$

die untere Grenze — ob angenommen oder nicht — von $\mid L_1 \cdots L_n \mid$ für die Punkte $\neq 0$ des Gitters des n-dimensionalen Euklidischen Raumes. Für beliebige Zahlen $\alpha_1, \ldots, \alpha_n$ vom selben Typ wie die Formen $L_1, \ldots, L_n$, also $\alpha_1, \ldots, \alpha_r$ reell, $\alpha_{r+1}, \ldots, \alpha_{r+s}$ komplex $\alpha_{r+s+j} = \overline{\alpha_{r+j}} (j = 1, \ldots, s)$ bezeichne

$$M_I(\alpha) = \min \mid (L_1 + \alpha_1) \cdots (L_n + \alpha_n) \mid,$$
$$M_P(\alpha) = \min \mid (L_1 + \alpha_1) \cdots (L_n + \alpha_n) \mid$$

unter der Nebenbedingung

$$L_k + \alpha_k > 0 \qquad (k = 1, \ldots, r) \tag{1}$$

und

$$M_S(\alpha) = \min \mid (L_1 + \alpha_1) \cdots (L_n + \alpha_n) \mid$$

unter der Nebenbedingung[1])

$$\Re(L_k + \alpha_k) > 0 \qquad (k = 1, \ldots, r + s).$$

Schließlich bezeichne M_I bzw. M_P bzw. M_S die obere Grenze von $M_I(\alpha)$ bzw. $M_P(\alpha)$ bzw. $M_S(\alpha)$ für alle Systeme $\alpha_1, \ldots, \alpha_n$ der angegebenen Art. Offenbar ist

$$M_I \leqq M_P \leqq M_S.$$

$c_1(n), \ldots, c_8(n)$ bezeichnen nur von n abhängige Konstanten; es ist möglich, doch wir sehen davon ab, diese Konstanten explizit anzugeben.

Im wesentlichen sind zwei Methoden bekannt (Čebotarev, Siegel), mit denen man zu allgemeinen Abschätzungen des Produktes inhomogener Linearformen gelangen kann.

1. Mit der Methode von Čebotarev sind folgende Abschätzungen bewiesen worden:

$$M_I \leqq 2^{-\frac{n}{2}} \mid D \mid \qquad (s = 0)^{2}), \tag{2}$$

$$M_I < c_1(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{s}{n-s}} \qquad (M_H \neq 0)^{3}), \tag{3}$$

---

[1]) $\Re\alpha$ bedeutet Realteil von α.

[2]) Vgl. [6], Satz 457.

[3]) Vgl. [1].

$$M_I < c_2(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{2s-1}{2n-1-2s}} \qquad (M_H \neq 0,\ s > 1)^{4)},$$

$$M_P \leqq \mid D \mid \qquad (s = 0)^{5)}, \tag{4}$$

$$M_P < c_3(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{s}{n-s}} \qquad (M_H \neq 0)^{6)}, \tag{5}$$

$$M_P < c_4(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{2s-1}{2n-1-2s}} \qquad (M_H \neq 0,\ s > 1)^{7)}. \tag{6}$$

Schreibt man in dem Beweis von (5) und (6) in [7] r, s an Stelle von r_1, r_2 und ersetzt man

$$Y_k + 1 > 0 \qquad (k = 1, \ldots, r)^{8)}$$

durch

$$\Re(Y_k + 1) > 0 \qquad (k = 1, \ldots, r + s)$$

und

$$Z_k + 1 > 0 \qquad (k = 1, \ldots, r)^{9)}$$

durch

$$\Re(Z_k + 1) > 0 \qquad (k = 1, \ldots, r + s)$$

und läßt man alles weitere ungeändert, so erhält man über (5) und (6) hinaus

Satz 1. *Für $M_H \neq 0$ ist*

$$M_S < c_5(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{s}{n-s}}; \tag{7}$$

für $M_H \neq 0$ und $s > 0$ ist

$$M_S < c_6(n) \mid D \mid \left(\frac{\mid D \mid}{M_H}\right)^{\frac{2s-1}{2n-1-2s}}.$$

2. Mit der Methode von Siegel sind bisher nur (2)[10] und (3)[11] (mit etwas anderen Konstanten) bewiesen worden. Hier soll gezeigt werden, daß sich auch (4), (5), (7) (erneut mit anderen Konstanten) mit dieser Methode beweisen lassen.

Zunächst beweisen wir

$$M_P < c_7(n) \mid D \mid \qquad (s = 0).$$

Es ist $r = n$;

$$R(x) = L_1^2(x) + \cdots + L_n^2(x) \text{ mit } x = (x_1, \ldots, x_n)$$

ist eine positiv-definite quadratische Form der Determinante $\Delta = D^2$. Für die sukzessiven Minima $r_1^2, \ldots, r_n^2$ einer solchen Form gilt bekanntlich

$$\Delta \leqq r_1^2 \cdots r_n^2 \leqq c_8(n)\, \Delta. \tag{8}$$

Aus $R(x) < r_n^2$ folgt die Beziehung

$$a_1 L_1(x) + \cdots + a_n L_n(x) = 0$$

mit reellen Zahlen $a_1, \ldots, a_n$; wir können

$$\max(\mid a_1 \mid, \ldots, \mid a_n \mid) = \mid a_n \mid > 0$$

---

4) Vgl. [5].

5) Vgl. [2], Satz 1.

6) Vgl. [7], Satz 6. 1. Hier sollen noch einige in [7] stehen gebliebene Druckfehler berichtigt werden: S. 559, Z. 22 v. o. lies komplex statt nichtreell; S. 461 und S. 462 lies in den Formeln stets Z statt z.

7) Vgl. [7], Satz 6. 2.

8) Vgl. [7], (6. 9).

9) Vgl. [7], (6. 13).

10) Eine Umarbeitung des Siegelschen Beweises einer Abschätzung von M_I ist in [3] veröffentlicht.

11) Vgl. [4], Satz 1 oder [5], (4).

annehmen. Dann folgt

$$L_1^2(x) + \cdots + L_{n-1}^2(x) \geqq \frac{1}{n} R(x).$$

Aus $R(x) < r_k^2$ folgt allgemein für eine geeignete Numerierung der Linearformen

$$L_1^2(x) + \cdots + L_{k-1}^2(x) \geqq \frac{k!}{n!} R(x).$$

Somit gibt es für jeden Gitterpunkt x ein k $(1 \leqq k \leqq n)$ mit

$$L_1^2(x) + \cdots + L_k^2(x) \geqq \frac{1}{n!} r_k^2.$$

Die quadratische Form

$$S(x) = r_1^{-2} L_1^2(x) + \cdots + r_n^{-2} L_n^2(x),$$

welche offenbar die Determinante

$$\Delta' = (r_1 \cdots r_n)^{-2} \Delta \tag{9}$$

hat, erfüllt also

$$S(x) \geqq \frac{1}{n!}$$

für alle Gitterpunkte x, d. h. es ist

$$s_1^2 \geqq \frac{1}{n!},$$

wenn $s_1^2, \ldots, s_n^2$ die sukzessiven Minima von $S(x)$ bezeichnen. Dann ist

$$s_1 + \cdots + s_n \leqq n s_n \leqq n (n!)^{\frac{n-1}{2}} s_1 \cdots s_n. \tag{10}$$

Wegen (8) und (9) ist

$$s_1^2 \cdots s_n^2 \leqq c_8(n)\, \Delta' \leqq c_8(n)\, (r_1 \cdots r_n)^{-2} \Delta \leqq c_8(n). \tag{11}$$

Ist nun $x^{(j)}$ ein Gitterpunkt mit

$$S(x^{(j)}) = s_j^2,$$

so ist

$$| L_k(x^{(j)}) | \leqq r_k s_j. \tag{12}$$

Wir definieren — abweichend von [3] — die reellen Zahlen η_j durch

$$\sum_{j=1}^{n} L_k(x^{(j)})\, \eta_j + \alpha_k = \frac{1}{2} \sum_{j=1}^{n} | L_k(x^{(j)}) | \qquad (k = 1, \ldots, n), \tag{13}$$

die Zahlen y_j als die größten ganzen Zahlen $\leqq (\eta_j + \frac{1}{2})$ und den Gitterpunkt

$$x = y_1 x^{(1)} + \cdots + y_n x^{(n)}.$$

Dann wird

$$L_k(x) = L_k(x^{(1)})\, y_1 + \cdots + L_k(x^{(n)})\, y_n.$$

Aus (13) folgt zunächst (1) und wegen (12) weiterhin

$$0 < | L_k(x)\, y_k + \alpha_k | \leqq \sum_{j=1}^{n} | L_k(x^{(j)}) | \leqq r_k (s_1 + \cdots + s_n) \qquad (k = 1, \ldots, n),$$

also wegen (10), (11), (8), $\Delta = D^2$ auch die gewünschte obere Abschätzung, was noch zu zeigen war.

Was den Beweis von (5) und (7) mit der Methode von Siegel betrifft, können wir uns kurz fassen. Der Kunstgriff, η_j durch (13) und nicht durch

$$\sum_{j=1}^{n} L_k(x^{(j)})\, \eta_j + \alpha_k = 0 \qquad (k = 1, \ldots, n)$$

wie in [3] zu definieren, ist auch in [4] wirksam. Definiert man also — in den Bezeichnungen von [4] — Θ_j durch

$$\sum_{j=1}^{n} \Theta_j \xi_p^{(j)} = \alpha_p + \frac{1}{2} \sum_{j=1}^{n} |\, \xi_p^{(j)} \,| \qquad (p = 1, \ldots, n)$$

und nicht durch[12])

$$\sum_{j=1}^{n} \Theta_j \xi_p^{(j)} = \alpha_p \qquad (p = 1, \ldots, n)$$

und beachtet man die daraus sich ergebenden geringfügigen Änderungen, so ergeben sich (5) und (7).

3. Die Anwendungen von Satz 1 auf die Theorie der algebraischen Zahlen liegen auf der Hand. Es sei $K = P(\Theta)$ eine endlich-algebraische Erweiterung des Körpers P der rationalen Zahlen. Unter den Konjugierten von Θ seien genau $\Theta^{(1)}, \ldots, \Theta^{(r)}$ reell und $\Theta^{(r+s+j)} = \overline{\Theta^{(r+j)}}$. In [7] und [8] war die *Signatur* der Zahl α aus K definiert als das System der r Zahlen sgn $\alpha^{(1)}, \ldots,$ sgn $\alpha^{(r)}$. Wir definieren hier die *Signierung* von α feiner durch das System der $(r+s)$ Zahlen sgn $\Re\alpha^{(j)} (j = 1, \ldots, r+s)$. Auf Grund von Satz 1 bleiben dann [7], Satz 6. 4 und [8], Hilfssatz 1 sowie die in [8] bewiesenen Sätze richtig, wenn man Signatur durch Signierung ersetzt.

Eine Zahl α aus K heißt *totalpositiv*, wenn sgn $\alpha^{(j)} = 1$ $(j = 1, \ldots, r)$ ist. Eine Zahl α aus K heiße *starkpositiv*, wenn sgn $\Re\alpha^{(j)} = 1$ $(j = 1, \ldots, r+s)$ ist. Man beachte jedoch, daß zwar das Produkt zweier totalpositiver Zahlen totalpositiv ist, während das Produkt zweier starkpositiver Zahlen im allgemeinen nur totalpositiv ist. Dagegen ist die Summe zweier starkpositiver Zahlen starkpositiv. Neben den totalpositiven Zahlen kommt daher auch den starkpositiven Zahlen eine ausgezeichnete Rolle in der additiven Zahlentheorie zu.

[12]) Vgl. [4], 39.

Literaturverzeichnis.

[1] *E. S. Barnes*, Note on non-homogeneous linear forms. Proc. Cambridge Phil. Soc. **49** (1953), 360—362.

[2] *J. H. H. Chalk*, The positive values of linear forms. Quart. J. Math. **18** (1947), 215—227.

[3] *H. Davenport*, Note on a result of Siegel. Acta arithm. **2** (1937), 262—265.

[4] *H. Davenport*, Linear forms associated with an algebraic number field. Quart. J. Math. (2) **3** (1952), 32—41.

[5] *H. Davenport*, On the product of n linear forms. Proc. Cambridge Phil. Soc. **49** (1953), 190—193.

[6] *G. H. Hardy* und *E. M. Wright*, An Introduction to the Theory of Numbers. Oxford 1954.

[7] *G. J. Rieger*, Über die Anzahl der Ideale in einer Idealklasse mod f eines algebraischen Zahlkörpers. Math. Annalen **135** (1958), 444—466.

[8] *G. J. Rieger*, Einige Sätze über Ideale in algebraischen Zahlkörpern. Math. Annalen **136** (1958), 339—341.

Eingegangen am 25. Mai 1959.

On the congruence properties of the zeta function of algebraic varieties *).

By *Bernard M. Dwork* in Baltimore (Maryland).

Let $\mathfrak{A}$ be a countable set and $\mathfrak{P}$ a partition of $\mathfrak{A}$ such that:

(i) If $\mathfrak{p} \in \mathfrak{P}$ then $\mathfrak{p}$ contains a finite number (written: $N\mathfrak{p}$) of elements.

(ii) For each given integer $s \geqq 1$, there exist only a finite number of $\mathfrak{p} \in \mathfrak{P}$ such that $N\mathfrak{p} = s$.

For each element $x \in \mathfrak{A}$, let the *degree* of x be defined to be $N\mathfrak{p}$, where $\mathfrak{p}$ is the element of $\mathfrak{P}$ to which x belongs. Let N_s be the number of elements of $\mathfrak{A}$ whose degree divides s. It is clear that

$$\exp\left(\sum_{s=1}^{\infty} N_s \frac{t^s}{s}\right) = \prod_{\mathfrak{p} \in \mathfrak{P}} \frac{1}{1 - t^{N\mathfrak{p}}}. \tag{1}$$

This power series shall be referred to as the partition function, $\zeta(\mathfrak{A}, \mathfrak{P}, t)$, or more briefly as $\zeta(\mathfrak{A}, t)$, if the partition, $\mathfrak{P}$, is clear. We shall view such power series as functions of a „p-adic variable". More precisely let p be a prime number and let Ω be the completion of the algebraic closure of the field of rational p-adic numbers. It follows from (1) that partition functions are power series with rational integral coefficients and therefore converge in the circle $|t| < 1$ in Ω.

Let k be a finite field and V an algebraic variety defined over k (regarded as a set of points in affine or projective space which are algebraic over k). V has a natural partition into conjugacy classes with respect to k. Since this partition will be understood throughout, the partition function may be denoted by $\zeta(V, t)$. If V is complete, irreducible and non-singular then $\zeta(V, t)$ is the zeta function of V as defined by Weil [1]. It will be assumed in the following that V is a complete projective varity. No assumption will be made concerning the irreducibility or non-singularity of V. Moreover our results may be carried over to affine varieties with very minor changes.

Let p be the characteristic of k and q the number of elements in k. It will be shown that $\zeta(V, t)$ is the ratio of two power series (with rational, p-integral coefficients) which converge in the circle $|t| < |q^{-1}|$ in Ω. Furthermore if V is a hypersurface then, excluding the pole at $t = 1$, the zeros (resp: poles) of $\zeta(V, t)$ must have p-adic value not less than $|q^{-1}|$ if V is of even (resp: odd) dimension. In particular if $\zeta(V, t)$ is rational then (in the ordinary sense) the zeros and poles are the reciprocals of algebraic integers, but if α is an algebraic integer, all of whose conjugates over the rationals have ordinary value less than q then α^{-1} cannot be a zero (resp: pole) of $\zeta(V, t)$ if V is a hypersurface of even (resp: odd) dimension (excluding a possible pole at $t = 1$).

*) This research was supported in part by National Science Foundation Grant Number G 3022.

An important consequence is that if V is an irreducible, non-singular hypersurface of even dimension with rational zeta function, then the first Betti number (as defined by Weil, [1]) is zero.

We are indebted to Professor Washnitzer for suggesting that we apply our p-adic results to this part of the Riemann Hypothesis.

Our method may be outlined briefly. Let K be a p-adic number field; f a polynomial in $n+1$ variables $X_0, \ldots, X_n$ with integral coefficients in K not all $\equiv 0 \bmod p$. Let V be the projective variety of characteristic p defined by $f \equiv 0 \bmod p$. It was noted by Warning, [2], that the number of rational points on V may be estimated by summing f^{q-1} over the set of points (in $(n+1)$-dimensional affine space of characteristic zero) whose coordinates are $(q-1)$-st roots of unity. We show (Lemma 3) that this sum is the trace of a linear transformation of $K[X] = K[X_0, \ldots, X_{n+1}]$. It will then follow that the partition function of V may be approximated in the p-adic sense by the characteristic polynomials of linear transformations of $K[X]$. (cf. Lemma 7, 8). The linear transformations themselves are defined and discussed in § 2, the geometric application being the subject of § 3. In § 4 we reverse our point of view and indicate what can be said about our linear transformations, if the Riemann Hypothesis were known to be true.

§ 1.

a. We show that the zeros and poles of a rational partition function are reciprocals of algebraic integers.

Let Z be the ring of integers and $Z\{t\}$ the ring of power series in one variable with coefficients in Z. Let L be a field of characteristic zero, and $L[t]$ the ring of polynomials with coefficients in L. Let Q be the field of rational numbers. The above assertion is a direct consequence of the following lemma.

Lemma 1. *If $H \in Z\{t\}$, and f and g are relatively prime elements of $L[t]$, such that $H = \frac{f}{g}$, then, aside from a common factor in L, f and g lie in $Z[t]$. Furthermore (after removal of the common factor) $g(0) = \pm 1$. In particular, if $H(0) = 1$ (as is the case for partition functions) then $f(0) = \pm 1$.*

Proof. Let $H = \Sigma a_i t^i$ and let $n = \max\{\text{degree } f, \text{ degree } g\}$. Then $Hg = f$ is equivalent to the existance in L of a solution of the infinite system of linear equations in $n+1$ variables,

$$a_s x_0 + a_{s-1} x_1 + \cdots + a_{s-n} x_n = 0, \quad s = n+1, n+2, \ldots.$$

Since these linear equations are defined over Q, the existence of a solution in L implies the existance of a rational, and therefore of an integral, solution. Hence there exist relatively prime polynomials f_1, g_1 in $Z[t]$ such that $H = \frac{f_1}{g_1}$. Hence by unique factorization in $L[t]$, there exists $\beta \in L$ such that $f_1 = \beta f$, $g_1 = \beta g$. Hence we may assume that f and g are relatively prime elements of $Z[t]$. There exist $a_0 \in Z$ and algebraic numbers $\alpha_1, \ldots, \alpha_n$ such that

$$g(t) = a_0 \prod_{i=1}^{n} (1 - \alpha_i t).$$

If α_1 is not an algebraic integer then there exists a prime $\mathfrak{p}$ in the splitting field of g such that $|\alpha_1|_{\mathfrak{p}} > 1$. Hence $H(t)$ converges p-adically at $t = \alpha_1^{-1}$, whence $f(\alpha_1^{-1}) = 0$, which

is impossible since $(f, g) = 1$ as elements of $Z[t]$. Hence $g(t) \in a_0(1 + tZ[t])$, whence a_0 divides f, which shows that $a_0 = \pm 1$. The assertion now follows with no difficulty.

b. Let $r > 0$ be an integer, and let T be the set of integers $\{0, 1, \ldots, r\}$. Let $\mathfrak{F}$ be the class of non-negative, real valued functions, η, on $T \times T$ such that

$$\sum_{x=0}^{r} \eta(x, z) = \sum_{y=0}^{r} \eta(z, y) \tag{2}$$

for all $z \in T$. Let c be an integer, $c \geqq 1$ and let h be the function on $T \times T$ defined by

$$h(x, y) = \begin{cases} x \text{ if } x + c > y, \\ 0 \text{ otherwise.} \end{cases}$$

We shall need the following estimate.

Lemma 2. *For each* $\eta \in \mathfrak{F}$

$$\sum_{P \in T \times T} h(P)\, \eta(P) \geqq c \sum_{P \in T \times T} \eta(P) - c \sum_{x=0}^{c-1} \sum_{y=0}^{2c-1} \eta(x, y), \tag{3}$$

where, in the second sum on the right, it is understood that $\eta(P) = 0$ *if* $P \notin T \times T$.

Proof. For each cyclically ordered subset

$$S = \{(a_i, b_i)\}_{i=0,1,\ldots,m} \quad (m \geqq 0),$$

of distinct points of $T \times T$ such that

$$b_i = a_{i+1},\ i = 0, 1, \ldots, m-1;\ b_m = a_0,$$

let $[S]$ denote the characteristic function of S and let $\mathfrak{F}'$ be the set of all such characteristic functions. We first show that given $\eta \in \mathfrak{F}$, there exist $\eta_1, \eta_2, \ldots, \eta_\lambda$ in $\mathfrak{F}'$ and positive real numbers, $e_1, e_2, \ldots, e_\lambda$, such that

$$\eta = \sum_{i=1}^{\lambda} e_i \eta_i.$$

Let (a, b) be a point in the carrier of η. It follows from (2) (letting $z = b$) that there exists $b' \in T$ such that (b, b') is in the carrier of η. Hence there exists a sequence, S', of points $\{(a_i, b_i)\}_{i=0,1,\ldots}$, in the carrier of η such that $b_i = a_{i+1}$, $i = 0, 1, \ldots$. Clearly S' contains only a finite number of distinct points and it may be assumed with no loss in generality that $(a_0, b_0) = (a_{m+1}, b_{m+1})$ for some integer $m \geqq 0$, and that $S'' = \{(a_i, b_i)\}_{i=0,1,\ldots,m}$ is a sequence of distinct points. But then $a_0 = a_{m+1} = b_m$, which shows that S'' is a sequence of the desired type. Let e be the minimal value assumed by η on S'', then $\eta - e[S''] \in \mathfrak{F}$, while its carrier contains fewer points than the carrier of η. The first assertion now follows by induction.

It is clear that it is enough to prove the lemma for $\eta \in \mathfrak{F}'$.

Let S be the cyclically ordered carrier of η and let R be the subset $y \geqq x + c$ of $T \times T$. Each point $P \in S$ is placed in one of three disjoint subsets of S according to the following three cases.

(i) P occurs as an element of a set $P_1, P_2, \ldots, P_{s+1}$ of consecutive points of S such that $P_j \in R$ for $j = 1, 2, \ldots, s$; $P_{s+1} \notin R$ and the element of S immediately preceeding P_1 does not lie in R.

(ii) P does not satisfy (i), P lies in the subset $x < c$ of $T \times T$.

(iii) P does not satisfy (i), P lies in the subset $x \geqq c$ of $T \times T$. It is clear that each element of $S \cap R$ falls in the first class. For a set $P_1, P_2, \ldots, P_{s+1}$ satisfying condition (i), let $P_j = (a_j, b_j)$, $1 \leqq j \leqq s+1$. We then have,

$$c \leqq b_1 = a_2 \leqq b_2 - c = a_3 - c \leqq b_3 - 2c$$
$$\leqq \cdots \leqq b_s - (s-1)\,c = a_{s+1} - (s-1)\,c,$$

whence

$$h(P_{s+1}) = a_{s+1} \geqq (s+1)\,c + (b_1 - 2c).$$

Therefore

$$\sum_{j=1}^{s+1} h(P_j)\ \eta(P_j) \geqq c \sum_{j=1}^{s+1} \eta(P_j) - \delta,$$

where

$$\delta = \begin{cases} c & \text{if } b_1 < 2c \\ 0 & \text{otherwise.} \end{cases}$$

However for $P_1 \notin R$,

$$b_1 < 2c \text{ implies } a_1 < c,$$

which explains the double sum on the right side of (3). The treatment of points in classes (ii) and (iii) is obvious and the lemma follows directly.

§ 2.

a. Let $L[X] = L[X_1, \ldots, X_n]$ be the ring of polynomials in n variables over an arbitrary field, L. Let $q > 1$ be a fixed integer, and let ψ be the endomorphism of $L[X]$ (as vector space over L)[1]) defined by

$$\psi\left(\prod_{i=1}^{n} X_i^{u_i}\right) = \begin{cases} \prod_{i=1}^{n} X_i^{\frac{u_i}{q}} & \text{if } q \mid u_i,\ i = 1, 2, \ldots, n, \\ 0 & \text{otherwise,} \end{cases} \tag{4}$$

for all n-tuples $(u_1, u_2, \ldots, u_n)$ with non-negative, integral coefficients. For $H \in L[X]$, $\psi \circ H$ denote the endomorphism $\xi \to \psi(H\xi)$ of $L[X]$. For $\xi \in L[X]$, the degree of the polynomial, $\psi(H\xi)$, is less than or equal to $\frac{1}{q} \cdot (\text{degree } H + \text{degree } \xi)$ and therefore

$$\text{degree } (\psi \circ H)\,\xi < \text{degree } \xi$$

if

$$\text{degree } \xi > \frac{1}{q-1} \cdot \text{degree } H.$$

For each non-negative integer, m, let L_m denote the finite dimensional subspace of all elements of $L[X]$ of degree not greater than m. It follows from the above that $\psi \circ H$ maps L_m into itself if

$$m \geqq m_0 = \left[\frac{1}{q-1} \cdot \text{degree } H\right],$$

and that the restriction, $(\psi \circ H)_m$, of $\psi \circ H$ to L_m is an endomorphism of L_m. Furthermore $\det(I - t(\psi \circ H)_m)$ is independent of m and likewise for the trace, $Tr((\psi \circ H)_m)^s$ for all integers $s > 0$, and all $m \geqq m_0$. We denote this common characteristic polynomial[2]) and

---

[1]) In this paper "endomorphism" is always in the module sense and never in the ring sense.

[2]) Contrary to customary terminology, we take the characteristic polynomial of a matrix, A, to be $\det(I - tA)$ instead of $\det(tI - A)$.

trace by $\det(I - t(\psi \circ H))$ and $Tr(\psi \circ H)^s$ respectively. Likewise $H \circ \psi$ is defined to be the endomorphism $\xi \to H\psi(\xi)$ of $L[X]$, its restriction to L_m is denoted by $(H \circ \psi)_m$ and $\det(I - t(H \circ \psi))$ is defined to be the polynomial $\det(I - t(H \circ \psi)_m)$ for all $m \geqq m_0 q$. It is well known that if Θ_1 and Θ_2 are two endomorphisms of a vector space of finite dimension then

$$\det(I - t\Theta_1 \circ \Theta_2) = \det(I - t\Theta_2 \circ \Theta_1).$$

Using this it is easy to show that

$$\det(I - t\psi \circ H) = \det(I - tH \circ \psi),$$

even though the endomorphism $\xi \to H\xi$ of $L[X]$ maps no finite dimensional subspace into itself.

b. Let L be algebraically closed and of characteristic zero. Let L^* be the multiplicative group of L and G the n-fold direct product of L^*. $L[X]$ has a natural imbedding in the space of all mappings of the (multiplicative) group G into L. It is easily verified that for $\xi \in L[X]$, $\psi(\xi)$ is the mapping of G into L given explicitly by

$$x \to q^{-n} \Sigma \xi(y), \tag{5}$$

the sum being over all $y \in G$ such that $y^q = x$. Thus $\psi(\xi)$ is determined at each point of G by taking an average over a coset of the group of elements of exponent q in G. We now show that ψ is also intimately associated with an average over points of G of exponent prime to q.

Lemma 3. *If $H \in L[X]$, s an integer, $s \geqq 1$ then*

$$(q^s - 1)^n \, Tr(\psi \circ H)^s = \Sigma H(x) \, H(x^q) \cdots H(x^{q^{s-1}}), \tag{6}$$

the sum being over the elements x of G of exponent $q^s - 1$.

Proof. We first reduce to the case $s = 1$.

$$(\psi \circ H)^s = \psi^s \circ H(X) \, H(X^q) \cdots H(X^{q^{s-1}}),$$

and ψ^s is the endomorphism of $L[X]$ obtained by replacing q by q^s in the definition of ψ. The reduction to $s = 1$ is now clear.

But $H \to Tr(\psi \circ H)$ and $H \to \Sigma H(x)$ (the sum being over the elements x of G of exponent $q - 1$) are homomorphisms of $L[X]$ (as L-modules) into L. Hence it may be assumed that H is a monomial $X^v = X_1^{a_1} X_2^{a_2} \cdots X_n^{a_n}$. But now $\Sigma H(x) = 0$ if $q - 1$ does not divide $v = (a_1, a_2, \ldots, a_n)$ and otherwise is $(q - 1)^n$. On the other hand $Tr(\psi \circ H)$ is the number of elements $w \in Z_+^n$ such that $\psi(X^w X^v) = X^w$, but this can happen if and only if $v = (q - 1)\, w$. This completes the proof.

c. Let Z_+ be the set of non-negative integers. For

$$u = (a_1, \ldots, a_n) \in Z^n,$$

let

$$\text{weight } u = \sum_{i=1}^{n} a_i.$$

For each integer $r \geqq 0$, let U_r be the set of elements of Z_+^n of weight not exceeding r.

Let R be an integral domain of characteristic zero. Let L be the field of fractions of R, and let d be a fixed integer $\geqq 1$. Let $Y = \{Y_u\}$ be a set of elements of R indexed by $U_{d(q-1)}$ and let

$$H(Y; X) = \sum_{u \in U_{d(q-1)}} Y^u X^u \in R[X] < L[X].$$

Let $q = p^c$, a power of a rational prime p and let

$$\Delta_s(t, Y) = \det (I - t\psi \circ H^{p^s}) \in Z[t, Y], \quad s = 0, 1, 2, \ldots.$$

Lemma 4. (i) *The coefficient of t^m in $\Delta_s(t, Y)$ is divisible by $q^{m-\varrho}$, where ϱ is the number of elements of U_{dq} (and therefore ϱ is independent of s).*

(ii) *If (Y') is another set of elements of R indexed by $U_{d(q-1)}$, then*

$$\Delta_s(t, Y) \equiv \Delta_s(t, Y + pY') \bmod p^{s+1} \tag{7}$$

in $Z[t, Y, Y']$, and furthermore

$$\Delta_{s+1}(t, Y) \equiv \Delta_s(t, Y^p) \bmod p^{s+1} \text{ in } Z[t, Y]. \tag{8}$$

Proof. Let

$$H_s(Y, X) = H(X)^{p^s}.$$

Clearly

$$H_s(Y + pY', X) \equiv H_s(Y, X) \bmod p^{s+1}$$

and equation (7) follows from the definitions. In the ring $Z[X, Y]$,

$$H_{s+1}(Y, X) \equiv H_s(Y^p, X^p) \bmod p^{s+1}$$

and therefore (ψ operates only on X),

$$\psi \circ H_{s+1}(Y, X) \equiv H_s(Y^p, X) \circ \psi \bmod p^{s+1}$$

as endomorphisms of the $Z[Y]$-module, $Z[X, Y]$. Using the remarks of § 2a,

$$\begin{aligned} \Delta_{s+1}(t, Y) &= \det (I - t\psi \circ H_{s+1}(X, Y)) \\ &\equiv \det (I - tH_s(Y^p, X) \circ \psi) \\ &= \det (I - t\psi \circ H_s(Y^p, X)) \\ &= \Delta_s(t, Y^p), \end{aligned}$$

which is equation (8).

The proof of (i) is more lengthy. Let

$$H_s(Y; X) = \Sigma A_v X^v, \quad A_v \in Z[Y],$$

the sum being over the set of all $v \in Z_+^n$ of weight not greater than $d(q-1)p^s$. Since

$$H_s(Y, X) \equiv H_{s-1}(Y^p, X^p) \bmod p^s,$$

it follows that A_v is divisible by p^{s-a}, $a = 0, 1, \ldots$ if v is not divisible by p^a. Let W be the set of all elements of Z_+^n of weight not greater than dp^s. For $u \in W$,

$$(\psi \circ H_s) X^u = \sum_{v \in W} C_{u,v} X^v, \text{ where } C_{u,v} = A_{qv-u}.$$

Let $\operatorname{ord} u = s - a$, where $a = s$ if q^s divides u and otherwise a is the smallest integer such that p^a does not divide u. In the notation of lemma 2, let T be the set $\{0, 1, \ldots, s\}$ and let h be the function on $T \times T$ as defined in § 1b. It follows from the above that for $(u, v) \in W \times W$,

$$h(\operatorname{ord} u, \operatorname{ord} v) \leqq \operatorname{ord} C_{u,v},$$

the highest power of p which divides $C_{u,v}$, in $Z[Y]$. The coefficient of t^m in $\Delta_s(t, Y)$ is zero if m exceeds the number of elements of W, and otherwise is a sum of products $\pm \prod_{i=1}^{m} C_{u_i, v_i}$ where $(v_1, v_2, \ldots, v_m)$ is a permutation of an ordered set $(u_1, \ldots, u_m)$ of m distinct elements of W. For one such product, let $\eta(a, b)$ be (for each $(a, b) \in T \times T$) the number of pairs $\{(u_i, v_i)\}_{1 \leq i \leq m}$ such that ord $u_i = a$, ord $v_i = b$. η certainly satisfies the conditions of Lemma 2, so that equation (3) is applicable. The left side of (3) is a lower bound on the power of p which divides the product in question and the first sum on the right hand side of (3) is clearly m. Finally $\sum_{a=0}^{c-1} \sum_{b=0}^{2c-1} \eta(a, b)$ does not exceed the number of $u \in W$ such that ord $u \leq c - 1$, which is the same as the number of elements of W divisible by p^{s-c} and this is not greater than ϱ. The lemma now follows from equation (3).

The principal ideal,

$$\mathfrak{a} = \left(\prod_{i=1}^{n} X_i\right)$$

of $L[X]$ is stable under ψ. Let $[\psi \circ H_s]$ denote the restriction of $\psi \circ H_s$ to $\mathfrak{a}$. The remarks of § 2a and of this paragraph apply to this mapping and in particular the coefficient of t^m in $\det(I - t[\psi \circ H_s])$ is divisible by $q^{m-\varrho'}$, where ϱ' is the number of elements of $(Z_+ - \{0\})^n$ of weight not exceeding dq.

Let now U be the set of elements of $(Z_+)^n$ of weight not greater than d, let $Y = \{Y_u\}$ be a set of elements of R indexed by U and let

$$F(Y; X) = \sum_{u \in U} Y_u X^u,$$

and let $H = F^{q-1}$ and let $\mathfrak{A}$ be the ideal in $Z[Y]$ generated by $\{Y_u^q - Y_u\}_{u \in U}$. Finally let

$$H_s = H^{q^s}, \; F_s = F^{q^s}.$$

Lemma 5. *If $q^v \leq m < q^{v+1}$, $m, v \in Z_+$, then*

$$\det(I - t\psi \circ H_{s+v}) \equiv \det(I - t\psi \circ H_s^m) \bmod (pq^s, \mathfrak{A})$$

in $Z[t, Y]$.

Proof. Since $F_{s+1}(Y, X) \equiv F_s(Y^q, X^q) \bmod pq^s$ in $Z[X, Y]$, we have as endomorphism of $Z[X, Y]$ as $Z[Y]$-module,

$$\begin{aligned} \psi \circ H_s^m &= \psi \circ F_s^{q^{v+1}-m} \circ F_{s+1}^{m-q^v} \\ &\equiv \psi \circ F_s(Y, X)^{q^{v+1}-m} \circ F_s(Y^q, X^q)^{m-q^v} \\ &= F_s(Y^q, X)^{m-q^v} \circ \psi \circ F_s(Y, X)^{q^{v+1}-m} \bmod pq^s, \end{aligned}$$

while

$$\psi \circ H_{s+v} = \psi \circ F_s^{q^v - q^v} \circ F_s^{q^{v+1}-m}.$$

The lemma follows from the earlier discussion of the effect of changing the order of mappings.

d. Let L, ψ and H be as in § 2a. Let S be the set $\{1, 2, \ldots, n\}$. For each non-empty subset A of S, let M_A be the monomial $\prod_{i \in A} X_i$ and let $L[X_A]$ be the ring of polynomials in the variables $\{X_i\}_{i \in A}$. Let T_A be the natural homomorphism of $L[X] = L[X_S]$ onto $L[X_A]$ given by $X_i \to 0$ for $i \notin A$, $X_i \to X_i$ for $i \in A$. Let Φ_A be the injection map of $L[X_A]$ into $L[X]$. If Θ is an endomorphism of the L-module $L[X]$ then $T_A \circ \Theta \circ \Phi_A$ is

an endomorphism of $L[X_A]$ which shall be written (A, Θ). In particular if Θ maps the ideal $M_A L[X]$ into itself then $M_A L[X_A]$ is stable under (A, Θ). In this case the restriction of (A, Θ) to $M_A L[X_A]$ is denoted by $[A, \Theta]$. $\psi \circ H$ has this property and thus $(A, \psi \circ H)$ is an endomorphism of $L[X_A]$ whose restriction to $M_A L[X_A]$ is denoted by $[A, \psi \circ H]$.

We shall need the following formal result.

Lemma 6.

$$\det (I - t\,\psi \circ H) = \Pi \det (I - t[A, \psi \circ H]) \tag{10}$$

the product being over all subsets A of S, it being understood that for A empty,

$$\det (I - t[A, \psi \circ H]) = 1 - tH(0, \ldots, 0).$$

Proof. Let d be the integer $\left[\frac{1}{q-1} \cdot \text{degree } H\right]$, W the space of polynomials of degree not greater than d. It is enough to consider the matrix relative to the monomials in W of the restriction of $\psi \circ H$ to W. A total ordering of the monomials in M is constructed so that for $m = 0, 1, \ldots, n-1$, the monomials divisible by exactly m variables preceed the monomials divisible by $m+1$ variables and such that for each subset A of S, if M and M' are exactly divisible by M_A (i. e., divisible by M_A but not by $X_i M_A$ for any $i \notin A$) then each monomial between M and M' is also exactly divisible by M_A. For each monomial M of W, we write

$$(\psi \circ H)\ (M) = \Sigma D(M, M')\, M',$$

the sum being over all monomials M' in W. It follows from the choice of the ordering that $D(M, M') = 0$ if M' preceeds M unless M and M' are exactly divisible by the same M_A. On the other hand the matrix $(D(M, M'))$ obtained by restricting M and M' to monomials exactly divisible by M_A is the same as the matrix of $[A, \psi \circ H]$ relative to the same basis. Thus the matrix for $\psi \circ H$ has a form consisting of the matrices of the transformations $[A, \psi \circ H]$ lying along the diagonal with zero elements lying below these. This completes the proof.

Corollary. *For each subset A of S*

$$\det (I - t[A, \psi \circ H]) = \Pi \det (I - t(B, \psi \circ H))^{(-1)^{m(A)-m(B)}}, \tag{11}$$

the product being over all subsets B of A, $m(B) + 1$ being the number of elements in B.

§ 3.

Let Ω be the completion of the algebraic closure of the rational p-adic numbers. Let $\mathfrak{O}$ be its ring of integers. In $\Omega\{t\}$, the ring of power series in one variable, let "weak" be used to denote the topology obtained by taking the set of all additive groups $t^m\Omega\{t\} + a\mathfrak{O}\{t\}$, $m = 1, 2, \ldots$; $a \in \Omega$, as a basis of the neighborhoods of zero and let "strong" be used to describe the topology obtained by taking the set of all additive groups, $a\mathfrak{O}\{t\}$, $a \in \Omega$ as a basis of the neighborhoods of zero. We note that in the class of power series which converge in the unit circle $|t| \leqq 1$ of Ω, the strong topology is the topology of uniform convergence on the unit circle.

We shall need the automorphism

$$\delta : h(t) \to \frac{h(t)}{h(qt)}$$

of the multiplicative group $1 + t\Omega\{t\}$. It is easily verified that

$$\delta^{-1}(h(t)) = \prod_{i=1}^{\infty} h(q^i t),$$

which certainly converges in the weak topology of $\Omega\{t\}$.

Let k be a finite field of characteristic p and of q elements. For $n \geqq 1$, let $\mathfrak{S}_n$ be n-dimensional projective space over k (viewed as containing only points algebraic over k). Let V be an algebraic variety in $\mathfrak{S}_n$ defined over k, i. e., V is the common intersection of a finite set of hypersurfaces $W_1, W_2, \ldots, W_r$ defined over k. Let $S^r = \{1, 2, \ldots, r\}$ and for each non-empty subset B of S_r, let $W_B = \bigcup_{i \in B} W_i$, which is again a hypersurface. By an obvious combinatorial argument, the partition function, $\zeta(V, t)$ is given explicitly by

$$\zeta(V, t) = \Pi\, \zeta(W_B, t)^{(-1)^{m(B)}} = \frac{\zeta(\mathfrak{S}_n, t)}{\Pi\, \zeta(\mathfrak{S}_n - W_B, t)^{(-1)^{m(B)}}} \tag{12}$$

where $m(B) + 1$ is the number of elements in B, and the product on the right is over all non-empty subsets, B, of S.

We may now restrict our attention to hypersurfaces. Let V be a hypersurface defined over k and let V' be the degenerate hypersurface defined by the equation $\prod_{i=0}^{n} X_i = 0$. For each integer $s \geqq 1$, let N_s be the number of points in $S_n - V \cup V'$ whose degree divides s. The partition function

$$\zeta(\mathfrak{S}_n - V \cup V', t) = \exp\left(\sum_{s=1}^{\infty} N_s \frac{t^s}{s}\right)$$

of $\mathfrak{S}_n - V \cup V'$ will be studied by considering the series

$$\zeta(\mathfrak{S}_n - V \cup V', t)^{-(-\delta)^{-n}} = \exp\left(-\sum_{s=1}^{\infty} \frac{N_s t^s}{s(q^s - 1)^n}\right). \tag{13}$$

Since

$$0 \leqq \frac{N_s}{(q^s - 1)^n} \leqq 1,$$

it follows that this series is of the form

$$1 + a_1 t + a_2 t^2 + \cdots,$$

where each coefficient a_i is rational, $-1 \leqq a_i \leqq 1$. It follows from the properties of partition functions and the automorphism, δ, that each a_i is p-integral.

Let K be the subfield of Ω, unramified over the rational p-adic numbers, whose residue class field is k. Let $f \in K[X_0, \ldots, X_n]$ be chosen with integral coefficients such that f mod p is the defining polynomial of the hypersurface V. Let

$$F_r = f^{(q-1)p^r}, \; r = 0, 1, \ldots.$$

Lemma 7. *The sequence of polynomials*

$$\det(I - t\psi \circ F_r)\}_{r=0,1,\ldots}$$

converges strongly in $\Omega\{t\}$ to a power series, $\langle f, t\rangle$, whose coefficients are rational numbers not greater than 1. $\langle f, t\rangle$ converges in the circle $|t| < |q^{-1}|$ in Ω and

$$\zeta(\mathfrak{S}_n - V \cup V') = \langle f, t\rangle^{-(-\delta)^n}. \tag{14}$$

Proof. Let G be the $(n+1)$-fold direct product of the multiplicative group of all points of Ω of finite order prime to p. Let $x \to x'$ be the natural map of G onto $\mathfrak{S}_n - V'$ obtained by reduction mod p composed with the usual map of affine $(n+1)$-space onto projective n-space. The restriction of this map to G_s, the subgroup of G consisting of points of exponent $q^s - 1$, is a $(q^s - 1)$ to one map onto the points of $\mathfrak{S}_n - V'$ whose degree over k divides s. Let

$$H_{r,s}(X) = \prod_{i=0}^{s-1} F_r(X^{q^i}).$$

For $x \in G_s$, we have

$$H_{r,s}(x) \equiv \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \bmod p^{r+1}, \text{ if } \begin{cases} x' \notin V, \\ x \in V. \end{cases}$$

Hence

$$N_s(q^s - 1) \equiv \sum_{x \in G_s} H_{r,s}(x) \bmod p^{r+1},$$

and therefore, using Lemma 3,

$$N_s \equiv (q^s - 1)^n \, Tr\,(\psi \circ F_r)^s \bmod p^{r+1}.$$

Thus in the weak topology of $\Omega\{t\}$,

$$\sum_{s=1}^{\infty} \frac{N_s t^s}{s(q^s - 1)} = \lim_{r \to \infty} \sum_{s=1}^{\infty} \frac{1}{s} \, Tr\,(\psi \circ F_r)^s t^s$$
$$= -\lim_{r \to \infty} Tr \log (I - t\,\psi \circ F_r).$$

It follows that the right side of equation (13) is the limit (which we denote by $\langle f, t \rangle$) in the weak topology of $\det\big(I - t(\psi \circ F_r)\big)$ as $r \to \infty$. It follows from equation (8) that if $q = p^c$, then with fixed integer $b \geqq 0$, the sequence of polynomials obtained by setting $r = b,\ b + c,\ b + 2c, \ldots$ converges strongly, and certainly the limit is $\langle f, t \rangle$. This proves the first assertion of the lemma. The first part of Lemma 4 shows that the coefficient of t^m in $\det\big(I - t(\psi \circ F_r)\big)$ is divisible by $q^{m-\varrho}$ in the ring of integers of K, where ϱ is independent of r. Hence the limit, $\langle f, t \rangle$, has the same property, and therefore, $\langle f, t \rangle$ converges for $|t| < |q^{-1}|$ in Ω. This completes the proof.

Corollary. *For each subset, A, of $S = \{0, 1, \ldots, n\}$ (notation of § 2d.) there exists power series $\langle A; f, t \rangle$, $[A; f, t]$ converging in the circle $|t| < |q^{-1}|$, such that*

$$\lim_{r \to \infty} \det\big(I - t(A, \psi \circ F_r)\big) = \langle A; f, t \rangle$$

and

$$\lim_{r \to \infty} \det\big(I - t[A, \psi \circ F_r]\big) = [A; f, t].$$

These power series are related by

$$\langle A; f, t \rangle = \Pi\,[B; f, t] \tag{15}$$

$$[A; f, t] = \Pi\,\langle B; f, t \rangle^{(-1)^{m(A) - m(B)}}, \tag{16}$$

the products being over the subsets of A.

Proof. The assertion concerning the limit of the characteristic polynomials of $(A); \psi \circ F_r)$ follows directly from the lemma, the corresponding property of $[A; \psi \circ F_r]$ is then a consequence of (11) and of the remark following the proof of Lemma 4. Equations (51 and (16) are direct consequences of (10) and (11) by passage to limits.

We may now deduce explicit formulae for the partition function of V in terms of limits of characteristic polynomials.

Lemma 8.

$$\zeta(\mathfrak{S}_n - V, t) = \Pi \langle A; f, t\rangle^{-(-\delta)^{m(A)}} = \Pi [A, f, q^{n-m(A)} t]^{-(-\delta)^{m(A)}} \tag{17}$$

$$\zeta(V, t) = \frac{\Pi [A, f, q^{n-m(A)} t]^{(-\delta)^{m(A)}}}{\prod_{i=0}^{n} (1 - tq^i)}, \tag{18}$$

the products being over subsets A of S.

Proof. (18) follows from (17), the fact that

$$\zeta(\mathfrak{S}_n, t) = \prod_{i=0}^{n} \frac{1}{1 - tq^i}$$

and the fact that

$$\zeta(\mathfrak{S}_n, t) = \zeta(V, t)\, \zeta(\mathfrak{S}_n - V, t),$$

since V and $\mathfrak{S}_n - V$ are disjoint. To prove (17), let $(\mathfrak{S}_n - V)_A$ be the subset of $\mathfrak{S}_n - V$ defined by

$$\prod_{i \in A} X_i \neq 0,$$

$$X_i = 0 \text{ for all } i \notin A.$$

From Lemma 7,

$$\zeta((\mathfrak{S}_n - V)_A, t) = \langle A, f, t\rangle^{-(-\delta)^{m(A)}}.$$

On the other hand

$$(\mathfrak{S}_n - V) = \bigcup (\mathfrak{S}_n - V)_A,$$

a disjoint union indexed by the subsets A of S. The first equality of (17) follows directly. The second equality of (17) is a direct consequence of (15).

Corollary. *For each hypersurface V, there exist power series, $H_1, H_2 \in 1 + tQ\{t\}$, with p-integral coefficients and which converge in the circle $|t| < |q^{-1}|$, such that*

$$(1 - t)\, \zeta(V, t) = [f, t]^{(-1)^n} \frac{H_1(qt)}{H_2(qt)} \tag{19}$$

where $[f, t] = [S; f, t]$.

The corollary is a direct consequence of (18), Lemma 7, the corollary of Lemma 7 and the properties of the δ-mapping.

It follows from (19) and (12) that the partition function of any variety is the ratio of two power series with rational, p-integral coefficients, which converge in the circle $|t| < |q^{-1}|$.

It follows from (19) that the partition function of a hypersurface of even dimension has a pole at $t = 1$, but no zero in the circle $|t| < |q^{-1}|$. For a hypersurface of odd dimension, there can be no pole other than $t = 1$ in the circle.

Suppose V is a hypersurface with rational partition function. We have seen that the zeros and poles of $\zeta(V, t)$ are the reciprocals of algebraic integers. Let α be an algebraic integer such that the geometric mean of the ordinary absolute magnitudes of the conjugates of α (over Q) is less than q. An elementary computation using the product formula for valuations in algebraic number fields shows that there exists a valuation of $Q(\alpha)$ above p at which $|\alpha^{-1}| < |q^{-1}|$. It follows that there exists an imbedding of $Q(\alpha)$ in Ω such that this inequality holds in the valuation of Ω. It follows from the hypothesis of rationality that α is zero or pole of $\zeta(V, t)$ in classical sense if and only if α is zero or pole in the p-adic sense. Hence α^{-1} cannot be a zero of $\zeta(V, t)$ if dimension V is even nor (if $\alpha \neq 1$) can α^{-1} be a pole of $\zeta(V, t)$ if dimension V is odd.

§ 4.

In the notation of § 3, if V is a non-singular, irreducible hypersurface in S_n, then according to the "topological" part of the Riemann Hypothesis, [1],

$$\zeta(V, t) \cdot \prod_{i=0}^{n-1} (1 - q^i t) = P_{n-1}(t)^{(-1)^n} \tag{20}$$

where P_{n-1} is a polynomial. It follows from (17) and (20) that if $f(X) \equiv 0 \bmod p$ is the defining equation of V (as in § 3) and if for each subset A of S containing three or more elements, the intersection V_A of V with the linear subspace $\{X_i = 0\}_{i \notin A}$ is irreducible and non-singular, then $\langle f, t\rangle$ is an entire function on Ω, i. e., a power series which converges everywhere. More explicitly, if we write $P_A(t)$ for the non-trivial factor of $\zeta(V_A, t)$ then (20) and (17) imply

$$\delta^n \langle f, t\rangle = (1 - t)^{1+\delta+\cdots+\delta^n} H(t) \, \Pi \, P_A(t) \tag{21}$$

where the product is over all subsets, A, of S which contain three or more elements and where $H(t)$ is a polynomial corresponding to the "missing" subsets, A, consisting of one or two elements. (That H is a polynomial, has no connection with the Riemann hypothesis; it may be shown by an elementary analysis of the homogeneous polynomials in one and two variables which may be obtained from $f(X)$ by setting the remaining variables equal to zero.)

If (21) is valid then the zeros of $\langle f, t\rangle$ form an infinite set, $\mathfrak{B}$, with the properties:

(i) If $\alpha \in \mathfrak{B}$ then $q^{-1}\alpha \in \mathfrak{B}$.

(ii) The zero's and poles of each of the varieties V_A lie in $\mathfrak{B}$; and furthermore each circle in Ω with center at the origin contains only a finite number of points lying in $\mathfrak{B}$.

Thus the Riemann hypothesis implies the validity in certain cases of the following conjecture: If f is a polynomial with coefficients in the ring of integers of a p-adic number field then $\langle f, t\rangle$ is an entire function on Ω. Independently of the Riemann hypothesis we note (using well known or easily proven facts about analytic functions on Ω) that if the polynomials

$$\Delta_r(t) = \det (I - t \, \psi \circ f^{(q-1)p^r})$$

converge uniformly on a circle, Γ', which properly contains the circle

$$\Gamma : |t| \leqq |q|^a \text{ in } \Omega,$$

for some fixed real number a, then $\langle f, t\rangle$ is the product of a polynomial h_a with integral rational p-adic coefficients whose roots lie in Γ and a power series H_a which converges on Γ but has no zeros in Γ. Furthermore h_a is the limit as $r \to \infty$ of the product of those factors $(1 - \lambda t)$ of $\Delta_r(t)$ which correspond to the roots of $\Delta_r(t)$ which lie in Γ. It follows from (15) and (18) that the non-trivial zeros and poles of $\zeta(V, t)$ which lie in Γ are either roots of h_a or are the product of such a root with a negative integral power of q and therefore the number of such zeros and poles of $\zeta(V, t)$ is finite.

It follows from § 3 that the remarks of the proceeding paragraph may be applied to the circle $|t| \leqq |q|^{-1+b}$ where b is any real number > 0 (the "b" is needed as Lemma 4 shows the polynomials $\Delta_r(t)$ to be uniformly bounded on the circle $|t| \leqq |q|^{-1}$, this together with weak convergence shows uniform convergence on the circle $|t| \leqq |q|^{-1+b}$).

Thus in particular we can conclude for any variety V (defined as in § 3 over the field of q elements) that given a real positive number, b, there exists a finite set of p-adic integers, $y_1, \ldots, y_r$, and square roots of unity, $\delta_1, \ldots, \delta_r$, such that

$$N_s \equiv y_1^s \delta_1 + \cdots + y_r^s \delta_r \bmod q^{(1-b)s - \log s} \tag{22}$$

where r depends on b (but not on s), log refers to the base q and N_s is the number of points of V rational over the field of q^s elements.

References.

[1.] *A. Weil*, Solutions of Equations in Finite Fields, Bull. Am. Math. Soc. **55** (1949), 497—508.

[2.] *E. Warning*, Bemerkung zur vorstehenden Arbeit von Herrn Chevalley, Abh. Math. Sem. Hamburg, **11** (1935), 76—83.

Eingegangen 19. Juni 1959.

Über die elementaren Beweise der Primzahlsätze.

Von *Hartmut Ehlich* in Tübingen.

1. 1949 bewies A. Selberg [11] elementar die asymptotische Beziehung

$$(1.1)\qquad \sum_{p \leq x} \log^2 p + \sum_{pq \leq x} \log p \log q = 2x \log x + O(x) \qquad (p, q \text{ Primzahlen}),$$

die sich nach Einführen der Funktionen $\vartheta(x) = \sum_{p \leq x} \log p$ und $\psi(x) = \sum_{n \leq x} \Lambda(n) = \sum_{m=1}^{\infty} \vartheta(\sqrt[m]{x})$ auch als

$$(1.2.)\qquad \vartheta(x) \log x + \sum_{p \leq x} \log p \; \vartheta\left(\frac{x}{p}\right) = 2x \log x + O(x),$$

bzw.

$$(1.3)\qquad \psi(x) \log x + \sum_{n \leq x} \Lambda(n) \; \psi\left(\frac{x}{n}\right) = 2x \log x + O(x)\ ^{1)}$$

schreiben läßt. Aus diesen Formeln, die heute als Selbergsche Formeln bezeichnet werden, haben P. Erdös [4] und A. Selberg [11] und dann weitere Autoren den Primzahlsatz

$$\pi(x) = \sum_{p \leq x} 1 = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right),$$

bzw. die dazu äquivalenten asymptotischen Gleichungen

$$(1.4)\qquad \vartheta(x) = x + o(x) \text{ oder } \psi(x) = x + o(x)$$

auf verschiedenen Wegen durch längere, elementare Schlußketten hergeleitet[2]).

Selberg[3]) vollzieht den Übergang von (1. 2) zu (1. 4) mittels eines Iterationsverfahrens, wobei er u. a. aus der elementaren Primzahltheorie die Mertenssche Formel

$$(1.5)\qquad \sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$$

heranzieht. Nach diesem Grundgedanken — Herleitung einer dem Problem angepaßten „Selbergschen Formel" und Heranziehung weiterer Ergebnisse der elementaren Primzahltheorie, möglichst der dem Problem entsprechenden Mertensschen Formel — konnten auch der Primzahlsatz für arithmetische Progressionen [12], [16], der Primidealsatz [13] und der Primzahlsatz für primitive binäre quadratische Formen [3] elementar bewiesen werden.

---

[1]) Eine kurze Herleitung von (1. 3) findet man in [7].

[2]) Zur Erdösschen Beweisanordnung vgl. [1], [2], [9].

[3]) Siehe auch [5], [6], [16], [17], [18].

Die in diesen Beweisen im Zusammenhang mit dem Iterationsverfahren, das fast immer den größten Teil der Arbeiten beansprucht, angestellten Überlegungen sind mühsam und werden im einzelnen unter Heranziehung der bei dem gegebenen Problem jeweils vorliegenden Tatsachen durchgeführt. Es ist daher wohl von Interesse, daß man diesen Teil der elementaren Beweise vereinfachen und vereinheitlichen kann, indem man die Gegebenheiten des speziellen Problems nicht ausnutzt und statt dessen nur das heranzieht, was wesentlich ist. Dieses Ziel erreichen die unten folgenden Sätze 1 und 2, die zugleich eine größere Übersichtlichkeit und, wenn man mehrere der oben genannten Sätze — z. B. in einer Vorlesung — herleiten will, eine wesentliche Abkürzung der elementaren Beweise erzielen; bei der Anwendung eines dieser Sätze sind jeweils nur die Voraussetzungen zu prüfen.

Satz 1. *Sind für die N Funktionen $f_\nu(x)$ $(\nu = 1, \ldots, N)$ die Ungleichungen*

$$\text{(A)} \qquad |f_\nu(x)| < K_1 x \quad \text{für } x > 0,$$

$$\text{(B)} \qquad |f_\nu(x') - f_\nu(x'')| < \frac{a}{N}(x' - x'') + K_2 \frac{x'}{\sqrt{\log x'}} \quad \text{für } x' \geqq x'' > 1, \ a > 0,$$

$$\text{(C)} \qquad |f_\nu(x)| < \frac{2}{N \log^2 x} \sum_{\lambda=1}^{N} \sum_{m \leqq x} \log m \left| f_\lambda\left(\frac{x}{m}\right)\right| + K_3 \frac{x}{\sqrt{\log x}} \quad \text{für } x > 1$$

erfüllt, gibt es ein $x_1 > 0$ und ein α_1 mit $0 < \alpha_1 < 1$, so daß für $x > x_1$ und $\nu = 1, \ldots, N$

$$\text{(D)} \qquad |f_\nu(x)| < \frac{a}{N} \alpha_1 x$$

ist, und folgt aus

$$\text{(E}_1\text{)} \qquad |f_\nu(x)| < \frac{a}{N} \alpha x \quad \text{für } x > x_0 \text{ und } 0 < \alpha < 1$$

stets

$$\text{(E}_2\text{)} \qquad \left| \sum_{m \leqq x} \frac{\log m}{m} f_\nu(m) \right| < \frac{a}{N} \alpha^2 x \log x + K_4 x \sqrt{\log x} \quad \text{für } x > \exp(\log^2 x_0),$$

so gibt es ein $c > 0$, so daß

$$\text{(F)} \qquad f_\nu(x) = O\left(x \exp\left(-c \frac{\log\log\log x}{\log\log\log\log x}\right)\right)^{4)},$$

sicher also

$$\text{(F}_1\text{)} \qquad f_\nu(x) = o(x)$$

ist.

Alle Voraussetzungen außer (D) ergeben sich für geeignete Funktionen — beim Primzahlsatz ($N = 1$) für die Funktion $f_1(x) = \vartheta(x) - x$ oder $f_1(x) = \psi(x) - x$, bei den anderen Sätzen für ganz analoge „Restfunktionen" — aus der dem behandelten Problem angepaßten Selbergschen Formel; (A) als triviale Folgerung, (B), (C), (E_1) und (E_2) durch einfache partielle Summation unter Benutzung von (1. 5) bzw. der damit äquivalenten Beziehung $\sum_{n \leqq x} \frac{\Lambda(n)}{n} = \log x + O(1)$. Die Voraussetzung (D) erhält man

---

4) P. Kuhn [8] zeigte unter Heranziehung weiterer Ergebnisse der elementaren Primzahltheorie, daß sich für $\psi(x) - x$ eine schärfere Abschätzung elementar gewinnen läßt. Mit Rücksicht auf möglichst wenige und einfache Voraussetzungen und übersichtliche Beweise wird in dieser Arbeit auf einen Ausbau dieser Methoden verzichtet.

ebenso leicht aus der Mertensschen Formel für das behandelte Problem, bei der sogar die Abschätzung $o(\log x)$ ausreicht. Steht diese nicht zur Verfügung — bei seinem Beweis des Primzahlsatzes für arithmetische Progressionen [12] scheint Selberg einen elementaren Beweis der Mertensschen Formel

$$\sum_{\substack{p \leq x \\ p \equiv L(Q)}} \frac{\log p}{p} = \frac{1}{\varphi(Q)} \log x + O(1), \quad (L, Q) = 1$$

nicht gekannt zu haben; für die durch eine primitive binäre quadratische Form dargestellten Primzahlen ist die Mertenssche Formel zwar bekannt, aber bis heute nicht elementar hergeleitet worden —, so ist zum Nachweis der Voraussetzung (D) eine längere Rechnung notwendig. Daß dabei die Anwendung eines der andern Primzahlsätze nicht notwendig ist, was der Absicht dieser Arbeit, eine Vereinheitlichung der Beweise zu erreichen, widersprechen würde, wurde in [3] gezeigt.

Etwas übersichtlichere Voraussetzungen hat Satz 2, der zugleich eine etwas schärfere Abschätzung liefert; seine Anwendung bei einem der oben genannten Probleme setzt jedoch die Kenntnis einer dem Problem angepaßten Mertensschen Formel (mit Restglied $O(1)$), aus der dann die Voraussetzung (D*) folgt, voraus.

Satz 2. *Sind für die Funktionen $f_\nu(x)$ ($\nu = 1, \ldots, N$) die Abschätzungen* (A), (B), (C) *und*

$$\text{(D*)} \qquad \left| \sum_{x' < m \leq x'} \frac{f_\nu(m)}{m^2} \right| < K_4^*$$

erfüllt, so ist

$$\text{(F*)} \qquad f_\nu(x) = O\left(x \left(\frac{\log\log\log x}{\log\log x}\right)^{1/2}\right).$$

2. Beweis von Satz 1. Im folgenden bezeichnen k, l, m, n, s natürliche, $a, c, \alpha, \alpha_1, \varrho, \sigma$ positive Zahlen und es sei stets $0 < \alpha < 1$ und $0 < \alpha_1 < 1$.

Hilfssatz 1. *Sind für die Funktionen $f_\nu(x)$ ($\nu = 1, \ldots, N$) die Beziehungen* (A), (B), (E_1) *und* (E_2) *mit $0 < \alpha < 1$ erfüllt und wird*

$$\text{(2. 1)} \qquad \begin{gathered} x_0^* = \exp(\log^2 x_0), \quad \gamma = (1-\alpha)^{-16} \\ K_5 = \operatorname{Max}\left\{1 + \frac{2N}{a} K_2, \quad 1 + \frac{N}{a} K_4\right\} \end{gathered}$$

gesetzt, so gibt es zu jedem y, das die Bedingungen

$$\text{(2. 2)} \qquad \begin{gathered} y > x_0^* \geqq 8, \quad y\alpha \geqq 2 \\ \frac{4K_5}{\alpha(1+3\alpha)} < \frac{32K_5}{\alpha(1-\alpha)} < \alpha\sqrt{\log y} < \frac{\gamma-1}{\gamma}\sqrt{\log y} < \sqrt{\log y} \end{gathered}$$

erfüllt, ein $y_\nu = y(f_\nu)$ aus dem Intervall $(y, \gamma y\rangle$, so daß für dieses y_ν aus (E_1) stets

$$\text{(2. 3)} \qquad |f_\nu(y_\nu)| < \frac{a}{N}\, \alpha\, \frac{1+3\alpha}{4}\, y_\nu \qquad (\nu = 1, \ldots, N)$$

folgt.

Beweis. Wegen $y(\gamma - 1) > y\alpha \geqq 2$ enthält $(y, \gamma y\rangle$ mindestens zwei ganze Zahlen. Wechselt $f_\nu(m)$ sein Vorzeichen in $(y, \gamma y\rangle$, so gibt es ein m', für das $f_\nu(m')$ und $f_\nu(m'+1)$ verschiedene Vorzeichen haben. Nach (B), (2. 1) und (2. 2) erhalten wir dann

$$|f_\nu(m')| \leqq |f_\nu(m'+1) - f_\nu(m')| < \frac{a}{N} + K_2 \frac{m'+1}{\sqrt{\log m'}} < \frac{a}{N} K_5 \frac{m'}{\sqrt{\log m'}} < \frac{a}{N}\, \alpha\, \frac{1+3\alpha}{4}\, m'.$$

Wechselt dagegen $f_\nu(m)$ sein Vorzeichen nicht in $(y, \gamma y\rangle$, so gibt es ein m' für das

$$0 \leqq \frac{|f_\nu(m')|}{m'} \sum_{y < m \leqq \gamma y} \log m \leqq \left| \sum_{y < m \leqq \gamma y} \frac{\log m}{m} f_\nu(m) \right|$$

gilt. Aus (E_1) folgt dann für dieses m' unter Beachtung von (E_2)

$$\frac{|f_\nu(m')|}{m'} y(\gamma - 1)(\log \gamma y - 1) \leqq \frac{a}{N} \alpha^2 y(\gamma + 1) \log \gamma y + 2 K_4 \gamma y \sqrt{\log \gamma y},$$

mittels (2. 2) also

$$\frac{|f_\nu(m')|}{m'} < \frac{a}{N} \alpha^2 \frac{\gamma + 1}{\gamma - 1} + \frac{2\left(2K_4\gamma + \frac{a}{N}(\gamma + 1)\right)}{(\gamma - 1)\sqrt{\log y}}$$

$$< \frac{a}{N} \alpha \frac{1 + 7\alpha}{8} + \frac{4 a \gamma K_5}{N(\gamma - 1)\sqrt{\log y}} < \frac{a}{N} \alpha \frac{1 + 3\alpha}{4}.$$

Hilfssatz 2. *Wird*

$$\beta = \frac{\alpha(1 - \alpha)}{32} < \frac{1}{4} \tag{2. 4}$$

gesetzt, so enthält unter den Voraussetzungen von Hilfssatz 1 das Intervall $(y, \gamma y\rangle$ ein Teilintervall $\langle z_\nu, z_\nu e^\beta)$, für dessen Punkte x sich aus (E_1)

$$|f_\nu(x)| < \frac{a}{N} \alpha \frac{1 + \alpha}{2} x \qquad (\nu = 1, \ldots, N) \tag{2. 5}$$

ergibt.

Beweis. Nach Hilfssatz 1 gibt es für jede Funktion f_ν ein y_ν aus $(y, \gamma y\rangle$, für das aus (E_1)

$$|f_\nu(y_\nu)| < \frac{a}{N} \alpha \frac{1 + 3\alpha}{4} y_\nu$$

folgt. Nach (B) ergibt sich dann mittels (2. 3) und $e^\beta < 1 + 2\beta < \frac{8 + 3\alpha + 5\alpha^2}{8 + 2\alpha + 6\alpha^2}$ für alle x aus $\langle y_\nu e^{-\beta}, y_\nu e^\beta\rangle$

$$|f_\nu(x)| < |f_\nu(y_\nu)| + \frac{a}{N} |y_\nu - x| + K_2 \frac{y_\nu e^\beta}{\sqrt{\log y_\nu}}$$

$$< \frac{a}{N} x \left(\frac{\alpha(1 + 3\alpha)}{4} e^\beta + e^\beta - 1 + \frac{N}{a} K_2 \frac{e^{2\beta}}{\sqrt{\log y}} \right)$$

$$< \frac{a}{N} x \left(\frac{3\alpha + 5\alpha^2}{8} + \frac{4K_5}{\sqrt{\log y}} \right) < \frac{a}{N} \alpha \frac{1 + \alpha}{2} x.$$

Da ferner nach (2. 4)

$$e^\beta - e^{-\beta} < 4\beta < \alpha < 1 - \frac{1}{\gamma},$$

also

$$y_\nu(e^\beta - e^{-\beta}) \leqq \gamma y(e^\beta - e^{-\beta}) < y(\gamma - 1)$$

ist, liegt entweder $\langle y_\nu e^{-\beta}, y_\nu)$ oder $\langle y_\nu, y_\nu e^\beta)$ ganz in $(y, \gamma y\rangle$.

Für den weiteren Beweis wählen wir x_1 so groß, daß $x_1 > x_0^*$ und mit $y = x_1$ und der in (D) auftretenden Zahl α_1 die Ungleichungen (2. 2) erfüllt sind. Ferner sei

$$K_6 = 2K_1 + 8\frac{a}{N} + K_3 \quad \left(\text{also: } K_6 \frac{N}{a} 2048 > 2\right). \tag{2. 6}$$

Wir definieren nun 4 Folgen durch die Rekursionsgleichungen

$$
\begin{aligned}
\alpha_{n+1} &= \alpha_n \left(1 - \frac{(1-\alpha_n)^3}{2048}\right) \\
\beta_n &= \frac{\alpha_n(1-\alpha_n)}{32} \\
x_n^* &= \exp(\log^2 x_n) \\
x_{n+1} &= \exp\left(\frac{2048 N K_6}{a\alpha_n(1-\alpha_n)^3}\right)^2 \log x_n^*.
\end{aligned}
\tag{2.7}
$$

Für die weitere Rechnung benötigen wir folgende Eigenschaften dieser Folgen:

(a) Es gibt drei positive Zahlen c_0, c_1 und M, so daß für alle $n > M$

$$c_0^{n-M} < \alpha_n < c_1^{n-M}$$

ist.

(b) Mit $y = x_n^*$ und $\alpha = \alpha_n$ sind alle Ungleichungen (2. 2) erfüllt.

(c) Es gibt ein $c > 0$, so daß für alle $n > M$

$$\alpha_n \leqq K_7 \exp\left\{-c\,\frac{\log\log\log x_n}{\log\log\log\log x_n}\right\}$$

ist.

Da $\{\alpha_n\}$ eine monoton fallende gegen Null konvergierende Zahlenfolge ist, gibt es ein M, so daß $\alpha_n < \frac{1}{2}$ für alle $n > M$ ist. Für diese n ist dann

$$\left(1 - \frac{1}{2048}\right) < 1 - \frac{(1-\alpha_n)^3}{2048} < \left(1 - \frac{1}{8\cdot 2048}\right).$$

Setzen wir $c_0 = \alpha_{M+2}$ und $c_1 = \left(1 - \frac{1}{8\cdot 2048}\right)$, so ist (a) für $n = M+1$ erfüllt. Ist nun $n \geqq M+1$, so erhalten wir aus $c_0^{n-M} < \alpha_n < c_1^{n-M}$ stets

$$\alpha_{n+1} = \alpha_n\left(1 - \frac{(1-\alpha_n)^3}{2048}\right)\begin{cases} < \alpha_n c_1 < c_1^{n+1-M} \\ > \alpha_n c_0 > c_0^{n+1-M}. \end{cases}$$

(b) ist nach der Wahl von x_1 und α_1 für $n = 1$ erfüllt; es genügt daher den Schluß von n auf $n+1$ zu zeigen. Nach Definition von x_n^* ist einerseits

$$x_{n+1}^*\alpha_{n+1} > \alpha_{n+1}\exp\left(\frac{2048 N K_6}{a\alpha_n(1-\alpha_n)^3}\right)^2 \log x_n^* > \alpha_{n+1}\exp 2\log x_n^* > \frac{\alpha_n}{2}\cdot 2x_n^* \geqq 2$$

und andererseits, da $\alpha_{n+1} > \frac{1}{2}\alpha_n$ ist,

$$
\begin{aligned}
\alpha_{n+1}^4(1-\alpha_{n+1})^2\log x_{n+1}^* &= \alpha_{n+1}^4(1-\alpha_{n+1})^2\log^2 x_{n+1} \\
&= \left(\frac{\alpha_{n+1}}{\alpha_n}\right)^4 \frac{(1-\alpha_{n+1})^2}{(1-\alpha_n)^{12}}\left(\frac{2048 N K_6}{a}\right)^4 \log^2 x_n^* \\
&> \frac{1}{2^4}\cdot 2^4 \log x_n^* > \alpha_n^4(1-\alpha_n)^2\log x_n^*.
\end{aligned}
$$

(c) erhalten wir ebenfalls unmittelbar aus den Definitionen von x_n und α_n, denn es ist mit $K_8 > 1$

$$
\begin{aligned}
\log\log\log x_n &= \log\log\left(\frac{2048 N K_6}{a}\right)^2 + \log\log\frac{1}{\alpha_n^2(1-\alpha_{n-1})^6} + \log\log\log^2 x_{n-1} \\
&= K_8 + \log\log\alpha_{n-1}^{-2}(1-\alpha_{n-1})^{-6} + \log\log\log x_{n-1} \\
&= (n-M-1)K_8 + \sum_{\mu=M+1}^{n-1}\log\log\alpha_\mu^{-2}(1-\alpha_\mu)^{-6} + \log\log\log x_{M+1} \\
&\begin{cases} > (n-M-1)K_8 > (n-M-1) \\ < (n-M)K_9 + \sum\limits_{\mu=1}^{n-M-1}\log\mu < (n-M)K_9'\log(n-M-1). \end{cases}
\end{aligned}
$$

Nach (a) ist dann für ein $c > 0$

$$-\frac{\log\log\log x_n}{\log\log\log\log x_n} > -(n-M)\,K_9' \geqq \frac{1}{c}\log c_1^{n-M} > \frac{1}{c}\log\alpha_n,$$

woraus (c) sofort folgt.

Wegen (c) ist — da $\exp\left(-c\,\frac{\log\log\log x}{\log\log\log\log x}\right)$ für $x > e^{e^{e^e}}$ monoton fällt — Satz 1 bewiesen, sobald wir noch zeigen:

Für $x > x_n$ ist $|f_\nu(x)| < \frac{a}{N}\,\alpha_n x \quad (\nu = 1, \ldots, N)$.

Da dies nach (D) sicher für $n = 1$ gilt, genügt es, den Induktionsschluß nachzuprüfen.

Aus (C) und (A) erhalten wir für $\nu = 1, \ldots, N$ und $x > x_n^*$

$$|f_\nu(x)| < \frac{2}{N\log^2 x}\sum_{\lambda=1}^{N}\left\{\sum_{m\leq\frac{x}{x_n^*}}\log m\left|f_\lambda\left(\frac{x}{m}\right)\right| + \sum_{\frac{x}{x_n^*}<m\leq x}\log m\left|f_\lambda\left(\frac{x}{m}\right)\right|\right\} + K_3\frac{x}{\sqrt{\log x}}$$

$$< \frac{2}{N\log^2 x}\sum_{\lambda=1}^{N}\sum_{m\leq\frac{x}{x_n^*}}\log m\left|f_\lambda\left(\frac{x}{m}\right)\right| + (2K_1 + K_3)\,x\sqrt{\frac{\log x_n^*}{\log x}}.$$

Im ganzen Intervall $\langle x_n^*, x\rangle$ können wir $f_\lambda\left(\frac{x}{m}\right)$ nach Induktionsannahme durch $\frac{a}{N}\,\alpha_n\frac{x}{m}$ abschätzen. Spalten wir jedoch das Integral $\langle x_n^*, x\rangle$ in die Teilintervalle $\langle x_n^*\gamma^{\mu-1}, x_n^*\gamma^\mu)$ mit $\gamma = (1-\alpha_n)^{-16}$ und $\mu = 1, \ldots, k = \left[\frac{\log\frac{x}{x_n^*}}{\log\gamma}\right]$ und $\langle x_n^*\gamma^k, x\rangle$ auf, so enthält — (E_1) ist mit $\alpha = \alpha_n$ erfüllt — nach Hilfssatz 2 jedes der k ersten Intervalle für jede Funktion f_λ ein Teilintervall $\langle z_{\lambda\mu}, z_{\lambda\mu}e^{\beta_n})$, in dem

$$\left|f_\lambda\left(\frac{x}{m}\right)\right| < \frac{a}{N}\,\alpha_n\frac{1+\alpha_n}{2}\,\frac{x}{m} \qquad (\lambda = 1, \ldots, N)$$

ist. Aus obiger Abschätzung folgt dann

$$\left|f_\nu\left(\frac{x}{m}\right)\right| < \frac{a}{N}\,\alpha_n x - \frac{a}{N}\,\alpha_n\frac{1-\alpha_n}{2}\,\frac{2x}{N\log^2 x}\sum_{\lambda=1}^{N}\sum_{\mu=1}^{k}\sum_{z_{\lambda\mu}\leq\frac{x}{m}\leq z_{\lambda\mu}e^{\beta_n}}\frac{\log m}{m}$$

$$+\left(2K_1 + 4\,\frac{a}{N} + K_3\right)x\sqrt{\frac{\log x_n^*}{\log x}},$$

also, wegen

$$\sum_{z_{\lambda\mu}\leq\frac{x}{m}\leq z_{\lambda\mu}e^{\beta_n}}\frac{\log m}{m} \geqq \beta_n\log\frac{x}{z_{\lambda\mu}e^{\beta_n}} \geqq \beta_n(k-\mu)\log\gamma,$$

$$|f_\nu(x)| < \frac{a}{N}\,\alpha_n x - \frac{a}{N}\,\alpha_n(1-\alpha_n)\,\beta_n\log\gamma\,\frac{k(k-1)}{2}\,\frac{x}{\log^2 x} + \left(2K_1 + 4\,\frac{a}{N} + K_3\right)x\sqrt{\frac{\log x_n^*}{\log x}}$$

$$< \frac{a}{N}\,\alpha_n x - \frac{a}{N}\,\frac{\alpha_n^2(1-\alpha_n)^2}{64\log(1-\alpha_n)^{-16}}\,x + \left(2K_1 + 8\,\frac{a}{N} + K_3\right)x\sqrt{\frac{\log x_n^*}{\log x}}$$

$$< \frac{a}{N}\,\alpha_n x\left(1 - \frac{\alpha_n(1-\alpha_n)^2}{1024\log\frac{1}{1-\alpha_n}}\right) + K_6 x\sqrt{\frac{\log x_n^*}{\log x}}.$$

Beachten wir noch, daß $\dfrac{\alpha_n}{\log \dfrac{1}{1-\alpha_n}} = \alpha_n \log^{-1}\left(1 + \dfrac{\alpha_n}{1-\alpha_n}\right) > 1 - \alpha_n$ ist, so erhalten wir für $x > x_{n+1}$

$$|f_\nu(x)| < \frac{a}{N}\alpha_n x\left(1 - \frac{(1-\alpha_n)^3}{1024}\right) + K_6 x \frac{a\alpha_n(1-\alpha_n)^3}{2048\,NK_6} = \frac{a}{N}\alpha_{n+1}x,$$

womit Satz 1 bewiesen ist.

3. Beweis von Satz 2. Satz 2 beweisen wir ähnlich wie Satz 1; wir müssen nur überall die Beziehungen (D), (E_1) und (E_2) durch (D*) ersetzen. Man beachte jedoch folgenden Unterschied: Die Hilfssätze 1 und 2 ergeben nur unter Annahme der Abschätzung (E_1) Auskunft über die Werte von f_ν, und (D) dient dazu, am Schluß des Beweises von Satz 1 den Induktionsanfang zu sichern; die folgenden Hilfssätze 3 und 4 ergeben dagegen, da die Ungleichung (D*) in ihre Voraussetzungen eingeht, sofort eine Abschätzung der Funktionen f_ν.

Hilfssatz 3. *Sind für die Funktionen $f_\nu(x)$ ($\nu = 1, \ldots, N$) die Ungleichungen* (A), (B) *und* (D*) *erfüllt, wird σ beliebig in $0 < \sigma < 1$ gewählt und*

$$\tau = \exp\frac{K_{10}}{\sigma}, \qquad K_{10} = \operatorname{Max}\left\{2\frac{N}{a}K_4^*, 2\right\} \tag{3.1}$$

gesetzt, so gibt es zu jedem y, das den Bedingungen

$$y > 8, \quad \sigma\sqrt{\log y} \geqq \operatorname{Max}\left\{\frac{N}{a}(K_2+1)+1,\ 3\frac{N}{a}K_2, K_{10}\right\} \tag{3.2}$$

genügt, für jede Funktion f_ν ein $\bar{y}_\nu = \bar{y}(f_\nu)$ aus dem Intervall $(y, \tau y\rangle$, so daß

$$|f_\nu(\bar{y}_\nu)| < \frac{a\sigma}{N}\bar{y}_\nu \tag{3.3}$$

ist.

Beweis. a) Wechselt $f_\nu(m)$ sein Vorzeichen in $(y, \tau y\rangle$, so erhalten wir — man vgl. die analoge Stelle im Beweis zu Hilfssatz 1 — für ein m' des Intervalls nach (B) und (3.2)

$$\begin{aligned}|f_\nu(m')| \leqq |f_\nu(m'+1) - f_\nu(m')| &< \frac{a}{N} + K_2\frac{m'+1}{\sqrt{\log m'}}\\ &< \frac{a}{N}\frac{m'}{\sqrt{\log y}}\left(1 + \frac{N}{a}(K_2+1)\right) \leqq \frac{a}{N}\sigma m'.\end{aligned}$$

b) Wechselt dagegen $f_\nu(m)$ sein Vorzeichen nicht in $(y\ \tau y\rangle$, so gibt es ein m', für das $\dfrac{|f_\nu(m')|}{m'} = \operatorname{Min}\left\{\dfrac{|f_\nu(m)|}{m};\ y < m \leqq \tau y\right\}$ und daher nach (D*)

$$\frac{|f_\nu(m')|}{m'}\sum_{y<m\leqq\tau y}\frac{1}{m} \leqq \left|\sum_{y<m\leqq\tau y}\frac{f_\nu(m)}{m^2}\right| < K_4^*,$$

nach (3.1) und (3.2) also

$$|f_\nu(m')| < m'\frac{K_4^*}{\log\tau - \dfrac{1}{y}} \leqq \frac{a}{N}\sigma m'$$

ist.

Hilfssatz 4. *Unter den Voraussetzungen des Hilfssatzes 3 gibt es für jede Funktion f_ν ein Teilintervall $(\bar{z}_\nu, \bar{z}_\nu e^{\frac{\sigma}{2}}\rangle$ von $(y, \tau y\rangle$, so daß für alle x aus diesem Teilintervall*

$$|f_\nu(x)| < 4\frac{a}{N}\sigma x \tag{3.4}$$

ist.

Beweis. Nach Hilfssatz 3 gibt es für jedes ν ein $\bar{y}_\nu$ in $(y, \tau y\rangle$, für das (3.3) erfüllt ist. Aus (B) folgt dann für alle x mit $\bar{y}_\nu e^{-\frac{\sigma}{2}} \leqq x \leqq \bar{y}_\nu e^{\frac{\sigma}{2}}$

$$\begin{aligned}|f_\nu(x)| &\leqq |f_\nu(\bar{y}_\nu)| + \frac{a}{N}|x-\bar{y}_\nu| + K_2\frac{\bar{y}_\nu e^{\frac{\sigma}{2}}}{\sqrt{\log y}}\\ &< \frac{a}{N}\sigma\bar{y}_\nu + \frac{a}{N}|x-\bar{y}_\nu| + K_2\frac{\bar{y}_\nu e^{\frac{\sigma}{2}}}{\sqrt{\log y}},\end{aligned}$$

woraus sich nach (3.2) und wegen $\left|1-\frac{\bar{y}_\nu}{x}\right| \leqq \exp\frac{\sigma}{2} - 1 < \sigma$

$$\begin{aligned}|f_\nu(x)| &< \frac{a}{N}\sigma x\left(\frac{\bar{y}_\nu}{x} + 1 + \frac{NK_2 e^{\frac{\sigma}{2}}}{a\sigma\sqrt{\log y}}\frac{\bar{y}_\nu}{x}\right)\\ &< \frac{a}{N}\sigma x\left(e^{\frac{\sigma}{2}} + 1 + \frac{NK_2 e^{\sigma}}{a\sigma\sqrt{\log y}}\right) < \frac{a}{N}\sigma x\left(3 + \frac{3NK_2}{a\sigma\sqrt{\log y}}\right) < 4\frac{a}{N}\sigma x\end{aligned}$$

ergibt.

Da nach (3.1) stets $\exp\frac{K_{10}}{\sigma} > \exp\sigma$ ist, kann nicht zugleich $\bar{y}_\nu e^{-\frac{\sigma}{2}} \leqq y$ und $\bar{y}_\nu e^{\frac{\sigma}{2}} \geqq \tau y$ gelten; eines der beiden Intervalle $(\bar{y}_\nu e^{-\frac{\sigma}{2}}, \bar{y}_\nu\rangle$ und $(\bar{y}_\nu, \bar{y}_\nu e^{\frac{\sigma}{2}}\rangle$ liegt daher ganz in $(y, \tau y\rangle$.

Von jetzt ab sei ξ_1 so groß, daß mit $\xi_1 = y$ und $\sigma = \frac{1}{2}$ die Ungleichungen (3.2) erfüllt sind, ferner sei

$$\begin{aligned}K_{11} &= \operatorname{Max}\left\{\frac{N}{a}K_1, 5\right\}, & K_{12} &= 4K_{11}\sqrt{K_{10}},\\ K_{13} &= 3\frac{a}{N}K_{11} + 2K_1 + K_3, & K_{14} &= 2\operatorname{Max}\left\{\frac{N}{a}K_{12}^2K_{13},\ 2K_{12}^3\right\}.\end{aligned} \tag{3.5}$$

Nun definieren wir — ähnlich wie in 2 — drei Folgen $\{\varrho_n\}$, $\{\sigma_n\}$ und $\{\xi_n\}$ durch die Rekursionsgleichungen

$$\begin{aligned}\varrho_{n+1} &= \varrho_n\left(1 - \frac{\varrho_n^2}{2K_{12}^2}\right), \quad \varrho_1 = K_{11}\\ \sigma_n &= \frac{\varrho_n}{2K_{11}}\\ \xi_{n+1} &= \exp\left(K_{14}^2\varrho_n^{-6}\log\xi_n\right),\end{aligned} \tag{3.6}$$

von denen wir folgende Eigenschaften herleiten:

Mit $y = \xi_n$ und $\sigma = \sigma_n$ sind die Ungleichungen (3.2) erfüllt, und ξ_n strebt gegen ∞. Ferner ist

$$\frac{1}{\sqrt{n}} < \varrho_n < \frac{K_{12}}{\sqrt{n}} \tag{3.7}$$

$$\varrho_n \leqq K_{15} \left(\frac{\log\log\log \xi_n}{\log\log \xi_n}\right)^{\frac{1}{2}}. \tag{3.8}$$

Da $\xi_n \geqq \xi_1 \geqq 8$ und

$$\sigma_n^2 \log \xi_n = \frac{\varrho_n^2}{4K_{11}^2} \frac{K_{14}^2}{\varrho_{n-1}^6} \log \xi_{n-1}$$

$$= \sigma_{n-1}^2 \frac{K_{14}^2 \varrho_n^2}{\varrho_{n-1}^8} \log \xi_{n-1} \geqq \sigma_{n-1}^2 \frac{K_{14}^2}{2K_{12}^6} \log \xi_{n-1} \geqq \sigma_{n-1}^2\, 2 \log \xi_{n-1},$$

also

$$\sigma_n^2 \log \xi_n \geqq \sigma_{n-1}^2 \log \xi_{n-1} \quad \text{und} \quad \xi_n \geqq 2\,\xi_{n-1}$$

ist, ist die erste Behauptung klar. (3. 7) ergibt sich leicht mittels Induktion; für $n = 1$ ist die Doppelungleichung nach Definition von K_{12} und ϱ_1 erfüllt, ist aber $n \geqq 1$, so folgt, da $u\left(1 - \frac{u^2}{2K_{12}^2}\right)$ in $0 \leqq u \leqq K_{11}$ monoton wächst,

$$\varrho_{n+1} = \varrho_n \left(1 - \frac{\varrho_n^2}{2K_{12}^2}\right) \begin{cases} < \dfrac{K_{12}}{\sqrt{n}}\left(1 - \dfrac{1}{2n}\right) < \dfrac{K_{12}}{\sqrt{n+1}} \\ > \dfrac{1}{\sqrt{n}}\left(1 - \dfrac{1}{2nK_{12}^2}\right) > \dfrac{1}{\sqrt{n+1}}. \end{cases}$$

Nach (3. 6) ist ferner — man beachte $\sigma_n < \sigma_1 = \frac{1}{2}$ —

$$\log\log \xi_n = \log\left(K_{14}^{2(n-1)} \prod_{\mu=1}^{n-1} \varrho_\mu^{-6} \log \xi_1\right)$$

$$= 2(n-1)\log K_{14} + 6 \sum_{\mu=1}^{n-1} \log \frac{1}{\varrho_\mu} + \log\log \xi_1$$

$$> 2(n-1)\log K_{14} + 6 \sum_{\mu=1}^{n-1} \log \frac{\sqrt{\mu}}{K_{12}} > 2(n-1)\log \frac{K_{14}}{K_{12}^3} > 2(n-1)\log 4,$$

und daher für $n \geqq 2$

$$\frac{\log\log \xi_n}{\log\log\log \xi_n} < \frac{1}{\log\log 4 \log n}\left(2n \log K_{14} + 3n \log n + \log\log \xi_1\right)$$

$$< K_{16} n \leqq K_{16} K_{12}^2 \varrho_n^{-2}.$$

Aus der letzten Abschätzung folgt aber (3. 8) unmittelbar.

Da $\left(\frac{\log\log\log x}{\log\log x}\right)^{\frac{1}{2}}$ für $x \geqq e^{e^e}$ monoton fällt, erhalten wir Satz 2 aus (3. 8) und der folgenden Behauptung:

Für $\nu = 1, \ldots, N$ *und* $n = 1, \ldots$ *ist*

$$|f_\nu(x)| < \frac{a}{N} \varrho_n x \quad \textit{für } x > \xi_n. \tag{3.9}$$

Auch (3. 9) beweisen wir mittels vollständiger Induktion. Nach (A) und der Wahl von ϱ_1 ist der Induktionsanfang gesichert, der Schluß von n auf $n+1$ ergibt sich wie folgt.

Nach (C) und (A) ist, wenn wir zur Abkürzung $u = \xi_n$ setzen, für $x > \xi_n$

$$|f_\nu(x)| < \frac{2}{N \log^2 x} \sum_{\lambda=1}^{N} \left\{ \sum_{m < \frac{x}{u}} \log m \left| f_\lambda\left(\frac{x}{m}\right)\right| + \sum_{\frac{x}{u} \leq m \leq x} \log m \left| f_\lambda\left(\frac{x}{m}\right)\right| \right\} + K_3 \frac{x}{\sqrt{\log x}}$$

$$< \frac{2}{N \log^2 x} \sum_{\lambda=1}^{N} \sum_{m < \frac{x}{u}} \log m \left| f_\lambda\left(\frac{x}{m}\right)\right| + (2K_1 + K_3)\, x \sqrt{\frac{\log u}{\log x}}.$$

Für $1 \leq m < \frac{x}{u}$, d. h. $u < \frac{x}{m} \leq x$ ist nach Induktionsvoraussetzung $\left|f_\lambda\left(\frac{x}{m}\right)\right| < \frac{a}{N} \varrho_n \frac{x}{m}$. Wir können aber f_λ noch besser abschätzen, denn setzen wir gemäß (3. 1) $\tau = \exp \frac{K_{10}}{\sigma_n}$ und spalten wir das Intervall $(u, x\rangle$ in die Intervalle

$$(u\tau^{\mu-1}, u\tau^\mu\rangle\,, \quad \mu = 1, \ldots, s = \left[\frac{\log \frac{x}{u}}{\log \tau}\right], \text{ und } (u\tau^s, x\rangle$$

auf, so enthält nach Hilfssatz 4 jedes der s Intervalle $(u\tau^{\mu-1}, u\tau^\mu\rangle$ für jede Funktion f_λ ein Teilintervall $(\bar{z}_{\lambda\mu}, \bar{z}'_{\lambda\mu}\rangle$ mit $\bar{z}'_{\lambda\mu} = \bar{z}_{\lambda\mu} \exp \frac{\sigma_n}{2}$, in dem $\left|f_\lambda\left(\frac{x}{m}\right)\right| < 4 \frac{a}{N} \sigma_n \frac{x}{m}$ ist. Daher erhalten wir

$$|f_\nu(x)| < \frac{a}{N} \varrho_n x - \frac{a}{N} \frac{2x}{N \log^2 x} (\varrho_n - 4\sigma_n) \sum_{\lambda=1}^{N} \sum_{\mu=1}^{s} \sum_{\bar{z}_{\lambda\mu} < \frac{x}{m} \leq \bar{z}'_{\lambda\mu}} \frac{\log m}{m}$$

$$+ \left(\frac{a}{N} K_{11} + 2K_1 + K_3\right) x \sqrt{\frac{\log u}{\log x}}$$

$$< \frac{a}{N} \varrho_n x - \frac{a}{N} \frac{x}{N \log^2 x} (\varrho_n - 4\sigma_n)\, \sigma_n \sum_{\lambda=1}^{N} \sum_{\mu=1}^{s} \log \frac{x}{\bar{z}'_{\lambda\mu}} + \left(\frac{a}{N} 2K_{11} + 2K_1 + K_3\right) x \sqrt{\frac{\log u}{\log x}}.$$

Für die Doppelsumme erhalten wir, wegen

$$\frac{x}{\bar{z}'_{\lambda\mu}} \geq \frac{x}{u\tau^\mu} \geq \frac{u\tau^s}{u\tau^\mu} = \tau^{s-\mu},$$

$$\sum_{\lambda=1}^{N} \sum_{\mu=1}^{s} \log \frac{x}{\bar{z}'_{\lambda\mu}} \geq N \log \tau \sum_{\mu=1}^{s} (s - \mu) = N \frac{s(s-1)}{2} \log \tau$$

$$\geq \frac{1}{2} N \left(\frac{\log^2 x}{\log^2 \tau} - \frac{5 \log x \log u}{\log^2 \tau}\right) \log \tau = \frac{1}{2} N \frac{\sigma_n}{K_{10}} (\log^2 x - 5 \log x \log u).$$

Daher folgt schließlich

$$|f_\nu(x)| < \frac{a}{N} \varrho_n x - \frac{a}{N} \frac{x}{2} (\varrho_n - 4\sigma_n) \frac{\sigma_n^2}{K_{10}} + \left(\frac{a}{N}\left(2K_{11} + \frac{3}{K_{10}}\right) + 2K_1 + K_3\right) x \sqrt{\frac{\log u}{\log x}}.$$

Für $x > \xi_{n+1}$ ist dann nach der Definition von u und ξ_{n+1} und (3. 5)

$$|f_\nu(x)| < \frac{a}{N} \varrho_n x \left(1 - \frac{(K_{11} - 2)\varrho_n^2}{8K_{11}^3 K_{10}}\right) + K_{12} x \sqrt{\frac{\log u}{\log \xi_{n+1}}}$$

$$< \frac{a}{N} \varrho_n x \left(1 - \frac{\varrho_n^2}{16 K_{11}^2 K_{10}}\right) + \frac{K_{13}}{K_{14}} \varrho_n^3 x$$

$$\leq \frac{a}{N} \varrho_n \left(1 - \frac{\varrho_n^2}{K_{12}^2}\right) x + \frac{a}{N} \frac{\varrho_n^3}{2K_{12}^2} x = \frac{a}{N} \varrho_{n+1} x.$$

Literaturverzeichnis.

[1] *R. Breusch*, Another proof of the prime number theorem. Duke Math. Journ. **21** (1954), 49—53.

[2] *J. G. van der Corput*, Démonstration élémentaire du théorème sur la distribution des nombres premiers. Math. Centrum Scr. **1** (1948).

[3] *H. Ehlich*, Ein elementarer Beweis des Primzahlsatzes für binäre quadratische Formen. Journ. f. Math. **201** (1959), 1—36.

[4] *P. Erdös*, On a new method in elementary number theory, which leads to an elementary proof of the prime number theorem. Proc. Nat. Acad. Sci. USA **35** (1949), 374—384.

[5] *E. K. Fogels*, An elementary proof of formulae of de la Vallée Poussin. Latvijas PSR Zinātnu Akad. Vestis **11** (1950), 123—130.

[6] *G. H. Hardy* und *E. M. Wright*, Einführung in die Zahlentheorie. Übersetzung der 3. Aufl., München 1958.

[7] *K. Iseki* und *T. Tatuzawa*, On Selberg's elementary proof of the prime number theorem. Proc. Japan Acad. **27** (1951), 340—342.

[8] *P. Kuhn*, Eine Verbesserung des Restgliedes beim elementaren Beweis des Primzahlsatzes. Math. Scand. **3** (1955), 75—89.

[9] *K. Prachar*, Primzahlverteilung. Berlin-Göttingen-Heidelberg 1957.

[10] *A. Selberg*, An elementary proof of Dirichlet's theorem about primes in an arithmetical progression. Annals of Math. **50** (1949), 297—304.

[11] *A. Selberg*, An elementary proof of the prime number theorem. Annals of Math. **50** (1949), 305—313.

[12] *A. Selberg*, An elementary proof of the prime number theorem for arithmetic progressions. Can. Journ. Math. **2** (1950), 66—78.

[13] *H. N. Shapiro*, An elementary proof of the prime ideal theorem. Comm. pure appl. Math. **2** (1949), 309—323.

[14] *H. N. Shapiro*, On a theorem of Selberg and generalisations. Annals of Math. **51** (1950), 485—497.

[15] *H. N. Shapiro*, On primes in arithmetical progressions I, II. Annals of Math. **52** (1951), 217—243.

[16] *W. Specht*, Elementare Beweise der Primzahlsätze. Berlin 1956.

[17] *E. Trost*, Primzahlen. Basel 1953.

[18] *E. M. Wright*, The elementary proof of the prime number theorem. Proc. Roy. Soc. Edinburgh (A) **63** (1952), 257—267.

Eingegangen 7. Juli 1959.

Ein metrischer Beitrag über Mahlersche S-Zahlen. I.

Von *Bodo Volkmann* in Mainz.

Für eine transzendente Zahl ξ bezeichnet man[1]) bei gegebenem natürlichem n mit $\vartheta_n(\xi)$ die obere Grenze der Menge der positiv-reellen Zahlen σ, für die es Folgen $p_1(x)$, $p_2(x), \ldots$ von verschiedenen Polynomen n-ten Grades mit ganzrationalen Koeffizienten so gibt, daß die Ungleichungen

$$(1) \qquad |p_i(\xi)| < \frac{1}{\|p_i\|^{n\sigma}} \qquad (i = 1, 2, \ldots)$$

erfüllt sind; wenn keine solchen σ existieren, sei $\vartheta_n(\xi) = 0$. Dabei ist $\|p_i\|$ die Höhe von $p_i(x)$. Als S-Zahlen im Sinne der Mahlerschen Klasseneinteilung bezeichnet man solche ξ, bei denen die Größe $\vartheta(\xi) = \sup\limits_{n=1,2,\ldots} \vartheta_n(\xi)$ positiv und endlich ist.

Wie sich mit Hilfe des Schubfachprinzips zeigen läßt[2]), gelten für alle komplexen, transzendenten ξ die Ungleichungen

$$(2) \qquad \vartheta_n(\xi) \geqq \frac{1}{2} - \frac{1}{2n} \qquad (n = 1, 2, \ldots),$$

und eine auf K. Mahler zurückgehende Vermutung besagt, daß hierin für fast alle[3]) ξ das Gleichheitszeichen gilt. Diese Aussage ist für $n = 1$ trivialerweise richtig; für $n = 2$ wurde sie von F. Kasch [1] bewiesen; für $n = 3$ ist nach einem kürzlich vom Verfasser [5] bewiesenen Satz bei fast allen ξ

$$(3) \qquad \vartheta_3(\xi) \leqq \frac{1}{2},$$

und für $n \geqq 4$ ist nach dem bisher schärfsten, auf F. Kasch und den Verfasser [2] zurückgehenden Ergebnis bei fast allen ξ

$$(4) \qquad \vartheta_n(\xi) \leqq \frac{3}{2} - \frac{2}{n}.$$

Daraus folgt insbesondere für fast alle ξ die Ungleichung

$$(5) \qquad \vartheta(\xi) \leqq \frac{3}{2},$$

die bereits von W. J. LeVeque [3] bewiesen worden war.

Im folgenden beweisen wir als Verschärfung von (4) und (5) den

Satz. *Für fast alle komplexen Zahlen ξ ist*

$$\frac{1}{2} - \frac{1}{2n} \leqq \vartheta_n(\xi) \leqq 1 - \frac{1}{2n} \qquad (n = 2, 3, \ldots).$$

---

[1]) Siehe z. B. [5].

[2]) Siehe z. B. Th. Schneider [4], S. 69.

[3]) Im Sinne des zweidimensionalen Lebesgueschen Maßes.

Folgerung *). *Für fast alle komplexen ξ ist $\vartheta(\xi) \leqq 1$.*

Beweis. Ein Polynom mit ganzrationalen Koeffizienten heiße *einfach*, wenn es nur einfache Nullstellen hat, und es sei $\tilde{\vartheta}_n(\xi)$ die analog $\vartheta_n(\xi)$, jedoch unter Beschränkung auf einfache Polynome $p(x)$ definierte Größe. Ferner seien $\sigma > 0$, $u \geqq 0$, $v > 0$ reelle Zahlen, $Q(u, v)$ die Menge der komplexen ξ mit $u \leqq Re\,\xi < u + 1$ und $v \leqq Im\,\xi < v + 1$, $Q^*(u, v)$ die $\frac{v}{2}$-Umgebung von $Q(u, v)$, $\tilde{S}_n(\sigma)$ die Menge der nichtreellen ξ mit $\tilde{\vartheta}_n(\xi) > \sigma$, sowie $\tilde{S}_n(\sigma, u, v) = \tilde{S}_n(\sigma) \cap Q(u, v)$. Für jedes $\xi \in \tilde{S}_n(\sigma, u, v)$ gibt es also eine Folge verschiedener, einfacher Polynome $p(x)$ vom Grade n mit (1).

Ein solches Polynom $p(x)$ (bei gegebenem ξ) sei fest gewählt, und seine Nullstellen seien $\alpha_1, \alpha_2, \ldots, \alpha_n$; ferner sei k die Anzahl derjenigen unter ihnen, die in $Q^*(u, v)$ liegen. Dann ist, falls $\|p\|$ genügend groß ist, sicher $1 \leqq k \leqq \left[\frac{n}{2}\right]$, da mit jeder Wurzel auch ihre Konjugierte vorkommen muß. Wir wählen nun die Numerierung so, daß die in $Q^*(u, v)$ liegenden Wurzeln $\alpha_1, \ldots, \alpha_k$ und die zu ihnen konjugierten $\alpha_{k+1}, \ldots, \alpha_{2k}$ sind; die (möglicherweise vorhandenen) übrigen Wurzeln bezeichnen wir auch mit $\beta_1, \ldots, \beta_{n-2k}$. Schließlich sei $q(x) = \prod_{j=1}^{k} (x - \alpha_j)$ und $\bar{q}(x) = \prod_{j=k+1}^{2k} (x - \alpha_j)$ gesetzt. Dann gilt, wenn allgemein $D(P)$ die Diskriminante eines Polynoms $P(x)$ und a_0 den höchsten Koeffizienten von $p(x)$ bezeichnet, wegen der vorausgesetzten Einfachheit von $p(x)$ die Ungleichung

$$1 \leqq \sqrt{|D(p)|} = |a_0|^{n-1} \sqrt{|D(q)\,D(\bar{q})|} \prod_{j=1}^{k} \prod_{l=k+1}^{2k} |\alpha_j - \alpha_l| \left\{ \prod_{j=1}^{2k} \prod_{l=1}^{n-2k} |\alpha_j - \beta_l| \prod_{\substack{j,l=1 \\ l<j}}^{n-2k} |\beta_j - \beta_l| \right\}.$$

Zur Abschätzung dieses Ausdrucks nach oben berücksichtigen wir, daß in dem Produkt in geschweiften Klammern jede Wurzel $\beta_\nu (\nu = 1, \ldots, n - 2k)$ in der Potenz $n - 1$ auftritt und daß die Beträge aller α_ν $(\nu = 1, \ldots, 2k)$ unterhalb einer nur von u und v abhängigen Konstanten[4]) C_1 liegen.

Daher wird nach einem auf N. I. Feldman zurückgehenden Hilfssatz[5]) jenes Produkt höchstens gleich $\left(\frac{C_1 \|p\|}{|a_0|}\right)^{n-1}$, so daß sich insgesamt, da $|D(q)| = |D(\bar{q})|$ ist, die Abschätzung

$$1 \leqq \sqrt{|D(p)|} \leqq |a_0|^{n-1} |D(q)| \cdot C_2 \left(\frac{\|p\|}{|a_0|}\right)^{n-1},$$

d. h.

$$|D(q)| \geqq \frac{C_3}{\|p\|^{n-1}} \tag{6}$$

ergibt. Ist nun α_1 eine Nullstelle mit $\min_{j=1,\ldots,k} |\xi - \alpha_j| = |\xi - \alpha_1|$, so gilt nach [2], Hilfssatz 4[6]) für $k \geqq 2$

$$|\xi - \alpha_1| \leqq C_4 \frac{\|q\|^{k-2}}{\sqrt{|D(q)|}} |q(\xi)|, \tag{7}$$

und hierin ist weiter

$$\|q\| \leqq \max\left(1, 2^k \max_{j=1,\ldots,k} |\alpha_j|^k\right) \leqq C_5, \tag{8}$$

[4]) $C_1, \ldots, C_{10}$ bedeuten positive, reelle Zahlen, die nicht von $\|p\|$ abhängen.

[5]) Siehe [2], Hilfssatz 2.

[6]) Dabei ist zu beachten, daß die dort angegebene Voraussetzung der Ganzzahligkeit aller Koeffizienten zum Beweis nicht benötigt wurde.

*) (Zusatz bei der Korrektur.) Wie der Verfasser inzwischen erfuhr, besitzt auch Herr *E. Wirsing* einen (unveröffentlichten) Beweis dieses Satzes. Vgl. auch die demnächst erscheinende Arbeit [6].

ferner auf Grund von (1)

$$(9)\qquad |q(\xi)| = \frac{|p(\xi)|}{|a_0| \prod\limits_{j=k+1}^{n} |\xi - \alpha_j|} \leqq \frac{\|p\|^{-n\sigma}}{|a_0|} \cdot \left(\frac{2}{v}\right)^{n-k} < C_6 \frac{\|p\|^{-n\sigma}}{|a_0|}.$$

Setzt man (6), (8), (9) in (7) ein, so erhält man

$$(10)\qquad |\xi - \alpha_1| < C_7 \frac{\|p\|^{\frac{n-1}{2} - n\sigma}}{|a_0|},$$

was offenbar auch in dem oben ausgeschlossenen Fall $k = 1$ gilt.

Die Menge aller einfachen Polynome $p(x) = a_0 x^n + \cdots + a_n$, $a_0 \neq 0$, mit wenigstens einer Nullstelle in $Q^*(u, v)$ werde mit $\mathfrak{P}_n(u, v)$ bezeichnet. Dann wird somit die Menge $\tilde{S}_n(\sigma, u, v)$ für jedes H_0 durch die Gesamtheit aller von den Zahlen ξ mit (10) gebildeten Kreise überdeckt, wenn α_1 alle in $Q^*(u, v)$ gelegenen Nullstellen aller Polynome $p(x) \in \mathfrak{P}_n(u, v)$ mit $\|p\| \geqq H_0$ durchläuft. Für das zweidimensionale Lebesguesche Maß μ erhält man daher die Ungleichung

$$\mu(\tilde{S}_n(\sigma, u, v)) \leqq \lim_{H_0 \to \infty} \sum_{H=H_0}^{\infty} \pi C_7^2 \left(\sum_{\substack{p(x) \in \mathfrak{P}_n(u,v) \\ \|p\| = H}} \frac{H^{n-1-2n\sigma}}{|a_0|^2} \right).$$

Zerlegt man die in Klammern stehende Summe in $n + 1$ Teilsummen mit der zusätzlichen Nebenbedingung $|a_0| = H$ bzw. $|a_1| = H, \ldots, |a_n| = H$, so folgt aus der Konvergenz der Reihe $\sum\limits_{|a_0|=1}^{\infty} \frac{1}{|a_0|^2}$, daß jede von ihnen kleiner als $C_8 H^{2n-2-2n\sigma}$ ist. Daher wird

$$\mu(\tilde{S}_n(\sigma, u, v)) \leqq C_9 \lim_{H_0 \to \infty} H^{2n-2-2n\sigma}$$

und somit $\mu(\tilde{S}_n(\sigma, u, v)) = 0$, falls $\sigma > 1 - \frac{1}{2n}$ ist.

Da sich $\tilde{S}_n(\sigma)$ als Vereinigung von abzählbar vielen Mengen der Form $\tilde{S}_n(\sigma, u, v)$ bzw. ihren Spiegelbildern bezüglich der reellen Achse darstellen läßt, z. B. wenn unabhängig voneinander u alle ganzrationalen und v alle positiven rationalen Zahlen durchlaufen, folgt für jedes $\sigma > 1 - \frac{1}{2n}$ auch $\mu(\tilde{S}_n(\sigma)) = 0$, d. h. für fast alle komplexen ξ ist

$$\tilde{\vartheta}_n(\xi) \leqq 1 - \frac{1}{2n} \qquad (n = 2, 3, \ldots).$$

Somit ist für fast alle komplexen ξ nach [2], Hilfssatz 5

$$\vartheta_n(\xi) \leqq \max_{\nu=1,\ldots,n} \vartheta(\xi) \leqq 1 - \frac{1}{2n} \qquad (n = 2, 3, \ldots),$$

wie behauptet.

Literaturverzeichnis.

[1] *F. Kasch*, Über eine metrische Eigenschaft der *S*-Zahlen. Math. Z. **70** (1958), 263—270.

[2] *F. Kasch* und *B. Volkmann*, Zur Mahlerschen Vermutung über *S*-Zahlen. Math. Ann. **136** (1958), 442—453.

[3] *W. J. LeVeque*, Note on *S*-numbers. Proc. Amer. Math. Soc. **4** (1953), 189—190.

[4] *Th. Schneider*, Einführung in die transzendenten Zahlen. Berlin-Göttingen-Heidelberg 1957.

[5] *B. Volkmann*, Zum kubischen Fall der Mahlerschen Vermutung. Math. Ann. **139** (1959), 87—90.

[6] *B. Volkmann*, Zur Mahlerschen Vermutung im Komplexen. Erscheint in den Math. Ann.

Eingegangen 16. Juli 1959.

Ein metrischer Beitrag über Mahlersche S-Zahlen. II.

Von *Friedrich Kasch* in Heidelberg.

1. In der vorhergehenden Arbeit mit dem gleichen Titel [4][1]) hat B. Volkmann unter Verwendung der in [2] gemeinsam entwickelten Schlußweise das folgende Resultat erhalten:

Für fast alle komplexen Zahlen ξ gilt:

$$\frac{1}{2} - \frac{1}{2n} \leqq \vartheta_n(\xi) \leqq 1 - \frac{1}{2n} \qquad (n = 2, 3, \ldots). \tag{1}$$

Darüber hinaus wurde in [1] und [3] gezeigt:

Für fast alle komplexen Zahlen ξ gilt

$$\vartheta_2(\xi) = \frac{1}{4}, \quad \vartheta_3(\xi) \leqq \frac{1}{2}. \tag{2}$$

Hier soll die rechte Seite von (1) zu

$$\vartheta_n(\xi) \leqq 1 - \frac{1}{n} \qquad (n = 2, 3, \ldots) \tag{3}$$

verschärft werden. Außerdem wird bewiesen, daß

$$\vartheta_4(\xi) \leqq \frac{5}{8} \tag{4}$$

für fast alle komplexen Zahlen ξ gilt.

2. Zum Beweis von (3) schließen wir zunächst wie in (I) bis zur Ungleichung[2])

$$1 \leqq \sqrt{|D(p)|} \leqq |a_0|^{n-1} |D(q)| C \left(\frac{\|p\|}{|a_0|}\right)^{n-1}.$$

Daraus ergibt sich

$$|D(q)| \geqq C \frac{\sqrt{|D(p)|}}{\|p\|^{n-1}},$$

und damit erhält man an Stelle von (10) in (I):

$$|\xi - \alpha_1| < C \frac{\|p\|^{\frac{n-1}{2} - n\sigma}}{|a_0| \, |D(p)|^{\frac{1}{4}}}.$$

Daraus folgt jetzt analog zur Schlußweise in (I):

$$\mu(\widetilde{S}_n(\sigma, u, v)) \leqq C \lim_{H_0 \to \infty} \sum_{H = H_0}^{\infty} \sum_{\substack{p(x) \in \mathfrak{P}_n(u,v) \\ \|p\| = H}} \frac{H^{n-1-2n\sigma}}{|a_0|^2 \sqrt{|D(p)|}}. \tag{5}$$

---

[1]) Im folgenden zitiert mit (I); aus (I) übernehmen wir alle Definitionen und Bezeichnungen.

[2]) Hier wie im folgenden bezeichne C eine positive reelle Zahl, die nicht von $\|p\|$ abhängt. Wir lassen zu, daß C an verschiedenen Stellen, wo es auftritt, verschiedene Werte besitzt.

Hieraus ergibt sich nach dem Vorbild von (I) die Behauptung (3), wenn die zweite rechts stehende Summe durch $CH^{2n-3-2n\sigma}$ nach oben abgeschätzt werden kann. Dazu zerlegen wir die Summe in $n+1$ Teilsummen mit der zusätzlichen Nebenbedingung $|a_0| = H$ bzw. $|a_1| = H, \ldots, |a_n| = H$, und es ist dann jede dieser Summen einzeln durch $CH^{2n-3-2n\sigma}$ abzuschätzen. Für die Summe mit $|a_0| = H$ ist das unmittelbar klar. Für die restlichen folgt die Behauptung sofort, wenn

$$\sum_{\substack{p(x) \in \mathfrak{P}_n(u,v) \\ \|p\| = H,\, a_0 \neq 0 \text{ fest} \\ |a_i| = H,\, a_i \text{ fest } (i > 0)}} \frac{1}{\sqrt{|D(p)|}} \leq CH^{n-2} \tag{6}$$

gezeigt werden kann. Dazu wird die Schlußweise aus [2] benutzt. Diese läßt sich allerdings nur anwenden, wenn feststeht, daß $D(p)$ in einem von a_0 und a_i verschiedenen Koeffizienten a_j ein Polynom mindestens 2. Grades ist. Dazu überlegen wir in Ergänzung von Hilfssatz 7 aus [2]:

Hilfssatz. *$D(p)$ ist als Polynom in a_n vom Grad $n-1$ und der Koeffizient von a_n^{n-1} ist bis auf das Vorzeichen gleich $n^n a_0^{n-1}$. $D(p)$ ist als Polynom in a_{n-1} vom Grad n und der Koeffizient von a_{n-1}^n ist bis auf das Vorzeichen gleich $(n-1)^{n-1} a_0^{n-2}$.*

Beweis: Bekanntlich gilt (Resultantendarstellung)

$$(-1)^{\binom{n}{2}} a_0 D(p) = \begin{vmatrix} a_0 & a_1 & \cdot & \cdot & \cdot & a_{n-1} & a_n & & & 0 \\ & a_0 & a_1 & \cdot & \cdot & \cdot & a_{n-1} & a_n & & \\ & & \cdot & & & & & \cdot & & \\ & & & \cdot & & & & & \cdot & \\ 0 & & & & \cdot & & & & & \cdot \\ & & & & & a_0 & a_1 & \cdot & \cdot & \cdot \\ na_0 & (n-1)\,a_1 & \cdot & \cdot & \cdot & a_{n-1} & & & & \\ & na_0 & (n-1)\,a_1 & \cdot & \cdot & \cdot & a_{n-1} & & & 0 \\ & & \cdot & & & & & \cdot & & \\ & & & \cdot & & & & & \cdot & \\ 0 & & & & \cdot & & & & & \cdot \\ & & & & & na_0 & (n-1)\,a_1 & \cdot & \cdot & \cdot \end{vmatrix} \begin{matrix} \\ \\ \\ \\ \\ \leftarrow (n-1)\text{-te} \\ \text{Zeile} \\ \\ \\ \\ \\ \\ \end{matrix}$$

Daraus liest man die erste Behauptung unmittelbar ab. Um die zweite einzusehen, subtrahiere man von der $(n-1+i)$-ten Zeile die i-te Zeile für $i = 1, \ldots, n-1$. Danach tritt a_{n-1} nur noch in den $n-1$ ersten und in der letzten Zeile auf. Der Wert der Determinante hat sich dabei nicht geändert, und man liest nun unmittelbar die zweite Behauptung ab.

Wir können nun den Beweis zu Ende führen. Zunächst kann $n \geqq 4$ angenommen werden, da für $n = 2$ und $n = 3$ die besseren Abschätzungen (2) gelten. Ferner folgt aus $p(x) \in \mathfrak{P}_n(u, v)$, daß $a_0 \neq 0$ ist. Dann läßt sich nach dem Hilfssatz stets ein a_j mit $j \neq 0$, $j \neq i$ so wählen, daß $D(p)$ als Polynom in a_j einen Grad > 2 besitzt. Seien nun in $D(p)$ alle a_k bis auf a_j fest gewählt. Summation über a_j liefert dann nach [2], Hilfssatz 6[3]) die Abschätzung

$$\sum_{\substack{a_j = -H \\ D(p) \neq 0}}^{H} \frac{1}{\sqrt{|D(p)|}} < C.$$

Da die von a_0, a_i, a_j verschiedenen $n-2$ Koeffizienten im Intervall $[-H, H]$ frei gewählt werden können, folgt (6).

---

[3]) *H.-E. Richert* verdanken wir einen sehr kurzen und einfachen Beweis von Hilfssatz 6, der auf der Hölderschen Ungleichung beruht.

3. Zum Beweis von (4) unterscheiden wir zwei Fälle, wobei wir auch jetzt die Bezeichnungen aus (I) zugrunde legen.

1. Fall. Nur eine Nullstelle α von $p(x) \in \mathfrak{P}_4(u, v)$ liegt in $Q^*(u, v)$. Dann gilt für $\xi \in Q(u, v)$ und die von α verschiedenen Nullstellen $\alpha_2, \alpha_3, \alpha_4$ von $p(x)$

$$\frac{v}{2} < |\xi - \alpha_i|.$$

Damit folgt

$$|\xi - \alpha| \left(\frac{v}{2}\right)^3 < \frac{|P(\xi)|}{|a_0|} \leqq \|p\|^{-4\sigma},$$

also

$$|\xi - \alpha| < C \|p\|^{-4\sigma}. \tag{7}$$

2. Fall. Zwei Nullstellen $\alpha = \alpha_1$ und α_2 von $p(x) \in \mathfrak{P}_4(u, v)$ liegen in $Q^*(u, v)$. Dann liefert die Schlußweise aus (I) (für $k = 2$, wobei keine β_i auftreten) die Abschätzung

$$\sqrt{|D(p)|} < C\, |a_0|^3\, |D(q)|.$$

Daraus folgt

$$|\xi - \alpha| < C \frac{|a_0|^{\frac{1}{2}} \|p\|^{-4\sigma}}{|D(p)|^{\frac{1}{4}}} \leqq C \frac{\|p\|^{\frac{1}{2}-4\sigma}}{|D(p)|^{\frac{1}{4}}}. \tag{8}$$

Man gelangt nun zu einer zu (5) analogen Abschätzung

$$\mu(\widetilde{S}_4(\sigma, u, v)) \leqq C \varliminf_{H_0 \to \infty} \sum_{H = H_0}^{\infty} \sum_{\substack{p(x) \in \mathfrak{P}_4(u,v) \\ \|p\| = H}} R(p),$$

wobei nach (7) und (8)

$$R(p) = \begin{cases} H^{-8\sigma} & \text{im 1. Fall} \\ \dfrac{H^{1-8\sigma}}{\sqrt{|D(p)|}} & \text{im 2. Fall} \end{cases}$$

zu setzen ist. Dann folgt unter Berücksichtigung von (6)

$$\sum_{\substack{p(x) \in \mathfrak{P}_4(u,v) \\ \|p\| = H}} R(p) \leqq \sum_{\substack{p(x) \in \mathfrak{P}_4(u,v) \\ \|p\| = H}} H^{-8\sigma} + \sum_{\substack{p(x) \in \mathfrak{P}_4(u,v) \\ \|p\| = H}} \frac{H^{1-8\sigma}}{\sqrt{|D(p)|}} \leqq C H^{4-8\sigma} + C H^{4-8\sigma} \leqq C H^{4-8\sigma}.$$

Daraus ergibt sich die Behauptung (4) analog zum allgemeinen Fall.

Literatur.

[1] *F. Kasch*, Über eine metrische Eigenschaft der S-Zahlen. Math. Z. **70**, 263—270 (1958).

[2] *F. Kasch* und *B. Volkmann*, Zur Mahlerschen Vermutung über S-Zahlen. Math. Ann. **136**, 442—453 (1958).

[3] *B. Volkmann*, Zum kubischen Fall der Mahlerschen Vermutung. Math. Ann. **139**, (1959), 87—90.

[4] *B. Volkmann*, Ein metrischer Beitrag über Mahlersche S-Zahlen. I. Journ. f. d. r. u. angew. Math. **203** (1959), 154—156.

Eingegangen 28. Juli 1959.

Tangents in Real Banach Spaces.

By *Günter Ewald* in Mainz and East Lansing (Michigan)
and *LeRoy M. Kelly* in East Lansing (Michigan).

1. Introduction.

The study of geometry in Minkowski or Banach spaces has been suggested by H. Busemann [3] as a necessary step for any progress in the theory of Finsler spaces[1]). As one of the basic concepts Busemann develops the "first curvature" of curves in Minkowski spaces and compares his (analytical) definition with definitions given by Menger, Alt, and Haantjes[2]) in general metric spaces. Although in the spaces under consideration the analytical definitions prove, in many respects, more useful than the metric ones, we will show in the present paper that the metric definitions of curvature enable us to discuss in a natural way the following question[3]): If we define the "free tangent" as the limiting position of an arbitrary secant of the curve at a point p (see section 2 for a precise definition), then evidently the existence of this tangent implies that the curve has neither a vertex nor a cusp at p, i. e. it is "curved" there[4]). Can this property be characterized by a metric condition related to curvature?

An example[5]) in the 2-dimensional Banach space with unit sphere $|x| + |y| = 1$ shows that even the existence of Alt, Menger or Haantjes curvature does not always insure the existence of a free tangent: the curve consisting of the union of the non-negative x and y axes possesses (vanishing) metric curvature at $(0, 0)$ but has no free tangent there. We will prove, however, that this phenomenon does not occur in any uniformly convex Banach space and that the following characterisations hold:

Let $K_M(a, b, c)$, $K_A(a, b, p)$, $K_H(a, b)$ be the expressions (see section 2) by means of which Menger, Alt, and Haantjes curvature are defined as

$$K_M(p) = \lim_{a,b,c\to p} K_M(a, b, c), \quad K_A(p) = \lim_{a,b\to p} K_A(a, b, p), \quad K_H(p) = \lim_{a,b\to p} K_H(a, b),$$

---

[1]) The theory of Finsler spaces and, more recently, that of spaces which are locally Banach (of arbitrary dimension) has indeed been developed considerably in the last decade, in particular on the basis of Busemann's ideas (papers by Busemann, Rund, Barthel, Laugwitz, Kawaguchi a. o.; for references see Rund [7]). — The results of this paper can be extended to spaces of the latter type.

[2]) See Blumenthal [1]. Another definition of metric curvature given later by Busemann himself we do not discuss here.

[3]) Apparently this question has not even been discussed so far in euclidean space.

[4]) Examples in the euclidean plane for which curvature (defined analytically in the usual way) does not exist: a) The curve $y = x^{1+\varepsilon}$, $1 > \varepsilon > 0$ at $(0, 0)$ (curvature "infinite"); b) the curve consisting of the union of the arcs $x^2 + (y-1)^2 = 1$, $x \geqq 0$; and $x^2 + (y-2)^2 = 4$, $x < 0$; at $(0, 0)$.

[5]) Due to L. M. Blumenthal, cf. I. J. Schoenberg [8], 964.

respectively, for p, a, b, c on an arc S in a uniformly convex Banach space, S being rectifiable in case of Haantjes curvature. Then S has a free tangent at p if and only if one of the following conditions is satisfied (xy denoting the distance between x and y):

$$\text{(I)} \qquad \lim_{a,b,c\to p} \{\sqrt{ab\cdot bc + bc\cdot ca + ca\cdot ac}\cdot K_M(a,b,c)\} = 0$$

$$\text{(II)} \qquad \lim_{a,b\to p} \{\sqrt{(ab)^2 + pa\cdot pb}\cdot K_A(a,b,p)\} = 0$$

$$\text{(III)} \qquad \lim_{a,b\to p} \{ab\cdot K_H(a,b)\} = 0.$$

Moreover, we will develop metric conditions for the existence of various types of (possibly one-sided) tangents.

2. Definitions and notation.

Let B be a real Banach space (of arbitrary dimension) with points $a, b, \ldots$ and (real-valued) norm $|a|$ of a. By ab we denote the "*distance*" $|a-b|$ of a and b. We use the following definitions some of which are valid for arbitrary metric spaces.

Definition 2. 1. An *arc* is a homeomorphic image of the real line. A *metric line* is an isometric image of the real line. One-dimensional subspaces of B are simply called *lines*.

Definition 2. 2. The point q is said to be *metrically between* points p and r provided $pq + qr = pr$.

Definition 2. 3. The point q is said to be *vectorially between* points p and r provided $q = \alpha p + (1-\alpha) r$ for some α satisfying $0 < \alpha < 1$ (α real).

Metric and vector betweenness need not coincide in general Banach spaces. Vector betweenness implies metric betweenness but not conversely.

Definition 2. 4. A Banach space is called *strictly convex* if metric and vector betweenness coincide (i. e. lines are metric lines and conversely). This is equivalent to requiring that no three points of the "unit sphere" be metrically collinear.

The requirement of definition 2. 4 prevents the spheres from having "flat spots" on them. In infinitely many dimensions, however, it doesn't prevent the spheres from "approaching flatness". To do this Clarkson [4] introduced the concept of uniformly convex Banach spaces:

Definition 2. 5. A Banach space is called *uniformly convex* if to each ε, $0 < \varepsilon \leqq 2$, there corresponds a $\delta(\varepsilon) > 0$ such that the conditions $|x| = |y| = 1$, $|x-y| \geqq \varepsilon$ imply $\left|\frac{x+y}{2}\right| \leqq 1 - \delta(\varepsilon)$.

Since an arc is a homeomorph of the real line it is, whenever the homeomorphism is specified, ordered and it makes sense to refer to the right and left sides of a point on the arc. If x is a point on the arc S, the symbol $\overrightarrow{S(x)}$ will refer to the portion of the arc to the right of the point x. An analoguous symbol is used on the left.

Definition 2. 6. An element r of a Banach space is called a *right hand unit tangent* to the arc S at the point p provided that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < pq < \delta$, $q \in \overrightarrow{S(p)}$ then $\left|\frac{p-q}{pq} - r\right| < \varepsilon$. The line determined by p and $p + r$ is called *right hand tangent* (*line*) to S at p. A similar definition holds on the left. If the left hand and the right hand unit tangents exist at a point and are negatives of

one another we say that the line defined by these two vectors at p is the *tangent* (*line*) to the arc at the point.

This is a slightly restricted definition of tangent but it will serve our purposes. — The existence of the left and right hand tangents is expressed by the existence of the corresponding unit tangents. This is also true for the following types of tangents:

Definition 2. 7. An elements r is called a *free right hand unit tangent* to S at $p \in S$, and the line determined by p and $p + r$ the *free right hand tangent* (*line*) to S at $p \in S$ provided that, given any $\varepsilon > 0$, there exists a $\delta > 0$ such that whenever x and y are different elements of $\overrightarrow{S(p)}$ with $px < \delta$ and $py < \delta$ then $\left|\frac{x-y}{xy} - r\right| < \varepsilon$. The definition may be extended in an obvious fashion to *left hand free unit tangent*, *left hand free tangent* (*line*) and *free tangent* (*line*).

Definition 2. 8. If q, r, s are any three pairwise distinct points, the number

$$K(q, r, s) = \frac{\sqrt{(qr + rs + sq)(qr + rs - sq)(qr - rs + sq)(-qr + rs + sq)}}{qr \cdot rs \cdot sq}$$

is called the curvature of the three points q, r, s. An arc has at a point p a *Menger curvature* $K_M(p)$ provided that for any three sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ of points of S approaching p with $a_i \neq b_i \neq c_i \neq a_i$, the sequence $\{K(a_i, b_i, c_i)\}$ has a finite limit.

Definition 2. 9. If in definition 2. 8, a_i be restricted to be p for each i, the resulting limit, if it exists, is called the *Alt curvature* at p and denoted by $K_A(p)$.

Thus if $K_M(p)$ exists, so does $K_A(p)$ and the two are equal, but not conversely.

Definition 2. 10. Let $l(q, s)$ denote the arc length of a rectifiable arc S with end points q and s. If we let $K_H(q, s) = \sqrt{\frac{4![l(q, s) - qs]}{l^3(q, s)}}$ and if $\{K_H(a_i, b_i)\}$ approaches a finite limit for any two sequences $\{a_i\}$, $\{b_i\}$ of points of S approaching the point $p \in S$, with $a_i \neq b_i$, we call this limit the *Haantjes curvature* $K_H(p)$ of S at p.

If in definition 2. 8 all distances are replaced by ordinary distances of the euclidean space associated with the Banach space we receive a concept of curvature ("first curvature") as included in a definition of n-th curvature given by E. Egerváry and G. Alexits [5], and considered by Busemann in [3] (in somewhat more general form). This curvature is equivalent to any one of those considered above provided they both exist. Our discussion could be extended to this curvature, too.

Throughout the paper the following notation is adopted: If p is a fixed point of S, and x any other point, the vector $p + \frac{x-p}{xp}$ is denoted $\bar{x}$. At some occasions p will be replaced by a variable point c_i. In general bars will be reserved for points on the unit sphere $(p, 1)$ (or $(c_i, 1)$, respectively).

In the sequel we will have frequent occasion to consider two sequences $\{a_i\}$ and $\{b_i\}$ of points (vectors) of S approaching p. Then we always assume (without mentioning it) that

$$a_i \neq b_i, \; p \neq a_i, \; p \neq b_i, \text{ for all } i.$$

If a third sequence $\{c_i\}$ occurs we assume that, furthermore,

$$a_i \neq c_i, \; b_i \neq c_i, \text{ for all } i \text{ (but not necessarily that } p \neq c_i).$$

In many instances, a_i and b_i will satisfy $pa_i \geqq pb_i$. In such circumstances we denote $p + \frac{a_i - p}{b_i p}$ by a_i', and $p + \frac{b_i - p}{a_i p}$ by $\hat{b}_i$. Thus a_i' will be a point (vector) outside, and $\hat{b}_i$ a point inside the sphere $(p, 1)$. Again p may be replaced by c_i.

By $\langle xy \rangle$ we denote the line determined by x and $y (x \neq y)$, and by $\overline{xy}$ the segment with endpoints x and y.

We use interchangeably the vector notation and the notation common to metric or affine geometry. The context should make clear which notation is in force.

3. The existence of a tangent.

From the examples given in section 1 two questions may arise: 1. Does the existence of any of the metric curvature mentioned imply at least the existence of a left or right hand unit tangent? 2. Does the existence of "infinite" curvature in uniformly convex Banach spaces always imply the existence of a tangent? Therefore we first present two examples giving a negative answer to both questions:

Example 3. 1. In the two-dimensional real affine plane with unit sphere $|x| + |y| = 1$ the series of horizontal segments joining the points $\left(\frac{1}{2^n}, \frac{1}{2^n}\right)$ to $\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$ and the series of vertical segments joining $\left(\frac{1}{2^{n+1}}, \frac{1}{2^n}\right)$ to $\left(\frac{1}{2^{n+1}}, \frac{1}{2^{n+1}}\right)$ together with reflections at the origin and the origin itself constitute an arc with zero curvature at each point but with neither a left nor a right hand tangent at the origin.

Example 3. 2. If the real numbers be metrized by calling $\sqrt{|x - y|}$ the distance between the two numbers x, y the resulting space is clearly a metric arc. Not so clear is the fact that it is congruent (isometric) to an arc in real separable Hilbert space. (See L. M. Blumenthal [2]). The curvature of this arc is at each point infinite and the arc is easily seen to have no right or left hand tangent at any point.

Theorem 3. 1. *A necessary condition for the existence of a right hand (unit) tangent at a point p of an arc S in a Banach space is that for any two sequences $\{a_i\}$ and $\{b_i\}$ of points of $\overrightarrow{S(p)}$ which approach p with $pa_i > pb_i$, the sequence $\left\{\frac{pa_i - a_i b_i}{pb_i}\right\}$ approaches one.*

Proof. Suppose the right hand unit tangent at p exists, and consider two sequences $\{a_i\}$ and $\{b_i\}$ of points of $\overrightarrow{S(p)}$, with $pa_i \geqq pb_i$, approaching p. From the definition of $\bar{a}_i, \bar{b}_i, a_i'$ we have $p\bar{a}_i = p\bar{b}_i = 1$; $a_i'\bar{b}_i = \frac{a_i b_i}{pb_i}$. (See fig. 1). Since the right hand unit tangent is assumed to exist $\{\bar{a}_i \bar{b}_i\}$ approaches zero. From

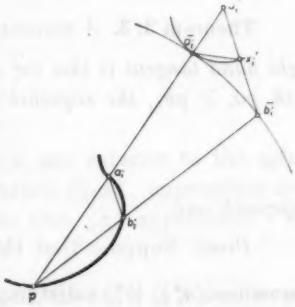


Fig. 1

$$\frac{pb_i + a_i b_i - pa_i}{pb_i} = \frac{p\bar{b}_i + a_i'\bar{b}_i - pa_i'}{p\bar{b}_i} = a_i'\bar{b}_i - a_i'\bar{a}_i < \bar{a}_i\bar{b}_i$$

(triangle inequality) it now follows that $\left\{\frac{pb_i + a_i b_i - pa_i}{pb_i}\right\}$ approaches zero or $\left\{\frac{pa_i - a_i b_i}{pb_i}\right\}$ approaches 1.

Example 3. 1 shows that this condition is not sufficient. However the following theorem holds:

Theorem 3. 2. *A sufficient condition for the existence of the right hand tangent at a point p of an arc S in a uniformly convex Banach Space is that for any two sequences* $\{a_i\}$ *and* $\{b_i\}$ *of points of* $\overrightarrow{S(p)}$ *approaching p with* $pa_i \geqq pb_i$, *the sequence*

$$\left\{\frac{pa_i - a_ib_i}{pb_i}\right\}$$

have 1 *as limit.*

Proof. We suppose the right hand tangent fails to exist. This implies the existence of a sequence $\{p_i\}$ of points of $\overrightarrow{S(p)}$ approaching p but such that $\{\bar{p}_i\}$ is not a Cauchy sequence. This in turn implies the existence of two subsequences of $\{p_i\}$, namely $\{a_i\}$ and $\{b_i\}$ with $pa_i \geqq pb_i$ and $\bar{a}_i\bar{b}_i > \varepsilon_0 > 0$ for all i and some fixed ε_0.

With the notation as in 3. 1 and with x_i' being the point on the segment $\overline{a_i'b_i}$ such that $a_i'x_i' = a_i'\bar{a}_i$ we note that

$$\frac{pb_i + a_ib_i - pa_i}{pb_i} = \frac{pb_i + a_i'\bar{b}_i - pa_i'}{p\bar{b}_i} = a_i'\bar{b}_i - a_i'\bar{a}_i = x_i'\bar{b}_i.$$

Now since $\left\{\frac{pa_i - a_ib_i}{pb_i}\right\}$ has 1 as limit, $\left\{\frac{pb_i + a_ib_i - pa_i}{pb_i}\right\} = \{x_i'\bar{b}_i\}$ has 0 as limit. It follows from the uniform convexity and from the fact that all points on the segment $\overline{\bar{a}_ix_i'}$ are exterior to the unit sphere $(p, 1)$ that $\{\bar{a}_ix_i'\}$ has zero as limit. The triangle inequality then compels $\{\bar{a}_i\bar{b}_i\}$ to have zero as limit, contrary to the definition of the subsequences $\{a_i\}$ and $\{b_i\}$. This contradiction establishes the theorem.

Corollary 3. 2. 1. *A necessary and sufficient condition for the existence of the right hand tangent at a point p of an arc S in a uniformly convex Banach space is that for any two sequences* $\{a_i\}$ *and* $\{b_i\}$ *of points of* $\overrightarrow{S(p)}$ *which approach p with* $pa_i \geqq pb_i$, *the sequence* $\left\{\frac{pa_i - a_ib_i}{pb_i}\right\}$ *approach* 1.

Theorem 3. 3. *A necessary condition that an arc S in a Banach space have at p a free right hand tangent is that for any two sequences* $\{a_i\}$, $\{b_i\}$ *of points of* $\overrightarrow{S(p)}$ *approaching p with* $pa_i \geqq pb_i$, *the sequence*

$$\left\{\frac{pa_i - pb_i}{a_ib_i}\right\}$$

approach one.

Proof. Suppose that the free right hand tangent exists but that for some pair of sequences $\{a_i^*\}$, $\{b_i^*\}$ satisfying the hypothesis of the theorem $\left\{\frac{pa_i^* - pb_i^*}{a_i^*b_i^*}\right\}$ fails to approach one. This implies the existence of a pair of subsequences $\{a_i\}$, $\{b_i\}$ of points of $\overrightarrow{S(p)}$ satisfying the hypothesis such that $\left\{\frac{pa_i - pb_i}{a_ib_i}\right\}$ is bounded away from one. Since $\frac{pa_i - pb_i}{a_ib_i} = \frac{pa_i' - p\bar{b}_i}{a_i'\bar{b}_i} = \frac{a_i'\bar{a}_i}{a_i'\bar{b}_i}$ this means that $\left\{\frac{a_i'\bar{a}_i}{a_i'\bar{b}_i}\right\}$ is bounded away from one.

Define y_i' to be the point on the line $\langle \bar{b}_i \bar{a}_i \rangle$ on the same side of $\bar{b}_i$ as is $\bar{a}_i$ such that $\bar{a}_i y_i' = \dfrac{\bar{a}_i \bar{b}_i}{\bar{a}_i a_i'}$, put $\bar{y}_i = p + \dfrac{y_i' - p}{y_i' p}$, and let z_i' be the point on the line $\langle \bar{a}_i \bar{b}_i \rangle$ on the same side of $\bar{a}_i$ as is $\bar{b}_i$, such that $\bar{a}_i z_i' = \dfrac{y_i' \bar{a}_i}{\bar{y}_i' \bar{y}_i}$ (see fig. 2). This implies, geometrically speaking, that $y_i' \bar{y}_i \bar{a}_i$ and $y_i' p z_i'$ are similar triangles. Thus we have

$$\frac{y_i' p}{y_i' \bar{y}_i} = \frac{p z_i'}{\bar{y}_i \bar{a}_i}, \quad \frac{y_i' p - y_i' \bar{y}_i}{y_i' \bar{y}_i} = \frac{p z_i'}{\bar{y}_i \bar{a}_i} - 1,$$

$$\frac{1}{y_i' \bar{y}_i} = \frac{p z_i'}{\bar{y}_i \bar{a}_i} - 1, \quad p z_i' = \bar{y}_i \bar{a}_i \left(1 + \frac{1}{y_i' \bar{y}_i}\right).$$

But $\left\{\dfrac{y_i' p}{1}\right\} = \left\{\dfrac{a_i' \bar{b}_i}{a_i' \bar{a}_i}\right\}$ which is bounded away from one and this means that $\{y_i' \bar{y}_i\}$ is bounded away from zero. Thus $\left\{1 + \dfrac{1}{y_i' \bar{y}_i}\right\}$ is bounded. From the existence of the free right hand tangent it follows that the line $\langle a_i' \bar{b}_i \rangle$ approaches a line $\langle p\bar{r} \rangle$ $(\bar{r} \in (p, 1))$. Hence the line $\langle p y_i' \rangle$ which is parallel to $\langle a_i' \bar{b}_i \rangle$ approaches the same line. This implies that $\{\bar{a}_i \bar{y}_i\}$, and therefore $\{p z_i'\}$, approaches zero. But since $\{\bar{a}_i \bar{b}_i\}$ approaches zero, the sequence $\{z_i'\}$ whose elements are on the lines $\langle \bar{a}_i \bar{b}_i \rangle$, is bounded away from p. This contradiction establishes the theorem.

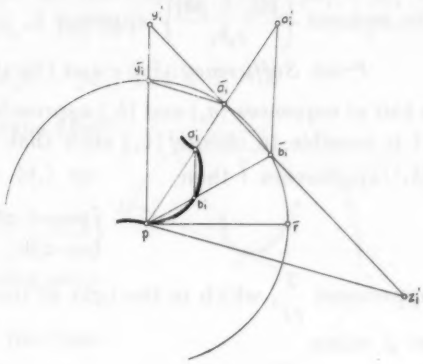


Fig. 2

Theorem 3. 4. *A sufficient condition that an arc S in a uniformly convex*[7]) *Banach space having at p a right hand tangent, have at p a free right hand tangent is that for any two sequences $\{a_i\}$ and $\{b_i\}$ of $\overrightarrow{S(p)}$ with $pa_i \geqq pb_i$, the sequence $\left\{\dfrac{pa_i - pb_i}{a_i b_i}\right\}$ have limit 1.*

Proof. Suppose r is the right hand unit tangent but the free right hand tangent fails to exist. This implies the existence of a pair of subsequences $\{a_i\}$ and $\{b_i\}$ of $\overrightarrow{S(p)}$ which approach p with $pa_i \geqq pb_i$ such that $\{\bar{y}_i\}$ ($\bar{y}_i$, y_i' same as in the proof of theorem 3. 3) is bounded away from $\bar{r} = p + r$. Then $\{\bar{y}_i \bar{r}\}$ is bounded away from zero.

Now for the sequences $\{a_i\}$ and $\{b_i\}$ we know that

$$\left\{\frac{pa_i - pb_i}{a_i b_i}\right\} = \left\{\frac{a_i' \bar{a}_i}{a_i' \bar{b}_i}\right\} = \left\{\frac{1}{y_i' p}\right\}$$

approaches one. Again, since all points of the segment $\overline{y_i' \bar{a}_i}$ are exterior to the sphere $(p, 1)$ it follows from uniform convexity that $\{y_i' \bar{a}_i\}$, and hence $\{\bar{y}_i \bar{a}_i\}$, approaches zero. From the existence of the right hand tangent we conclude that $\{\bar{a}_i\}$ approaches $\bar{r}$ and hence $\{\bar{y}_i \bar{r}\}$ approaches zero. This is a contradiction.

Corollary 3. 4. 1. *A necessary and sufficient condition that an arc S in a uniformly convex Banach space have at p a right hand free tangent is that for any pair of sequences $\{a_i\}$ and $\{b_i\}$ of $\overrightarrow{S(p)}$ with $pa_i \geqq pb_i$, the sequences $\left\{\dfrac{pa_i - pb_i}{a_i b_i}\right\}$ and $\left\{\dfrac{pa_i - a_i b_i}{pb_i}\right\}$ approach 1.*

---

[7]) Theorem 3. 4 still holds if the Banach space is *locally* uniformly convex in the sense of A. R. Lovaglia [6].

Theorem 3. 5. *If the right hand and left hand tangents exist at a point p of an arc S in a strictly convex Banach space then a necessary and sufficient condition that S have at p a tangent is that for any two sequences $\{a_i\}$ and $\{b_i\}$ approaching p, $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$, the sequence $\left\{\frac{pa_i + pb_i}{a_i b_i}\right\}$ approach 1.*

Proof. Sufficiency: Let r and l be the right and left hand unit tangents and consider a pair of sequences $\{a_i\}$ and $\{b_i\}$ approaching p, $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$. Put $\bar{r} = p + r$, $\bar{l} = p + l$. It is possible to choose $\{b_i\}$ such that $pa_i = pb_i$ for all i. Since $\{\bar{a}_i\}$ approaches r and $\{\bar{b}_i\}$ approaches $\bar{l}$ then

$$\left\{\frac{pa_i + pb_i}{a_i b_i}\right\} = \left\{\frac{p\bar{a}_i + pb_i}{\bar{a}_i \bar{b}_i}\right\}$$

approaches $\frac{2}{\bar{r}\bar{l}}$, which in the light of the hypothesis is 1. Hence $r = -l$ and the tangent at p exists.

Necessity: Suppose $r = -l$ but $\left\{\frac{pa_i + pb_i}{a_i b_i}\right\}$ is bounded away from 1 for a pair of sequences $\{a_i\}$ and $\{b_i\}$, approaching p, $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$, with $pa_i \geqq pb_i$.

Note now that $|pa_i' - a_i'\bar{b}_i| \leqq 1$. The sequence $\{pa_i'\}$ must be bounded since otherwise for some subsequence $\{a_{i_k}\}$, the sequence $\{a_{i_k}'\bar{b}_{i_k}\}$ would be unbounded, and

$$\left\{\frac{pa_{i_k} + pb_{i_k}}{a_{i_k} b_{i_k}}\right\} = \left\{\frac{pa_{i_k}' - a_{i_k}'\bar{b}_{i_k}}{a_{i_k}'\bar{b}_{i_k}} + 1 + \frac{p\bar{b}_{i_k}}{a_{i_k}'\bar{b}_{i_k}}\right\}$$

would approach 1. Let a_i^* be the point on the line $\langle p\bar{r}\rangle$ such that $pa_i^* = pa_i'$ and that a_i^* is on the same side of p as is $\bar{r}$. Then $\{pa_i^*\}$ is bounded and hence there exists a subsequence $\{a_{i_l}\}$ of $\{a_i\}$ such that $\{a_{i_l}^*\}$ converges to a point a^* of the line $\langle p\bar{r}\rangle$. We have

$$\frac{a_{i_l}' a_{i_l}^*}{\bar{a}_{i_l}\bar{r}} = \frac{pa_{i_l}^*}{1}, \quad a_{i_l}' a_{i_l}^* = (\bar{a}_{i_l}\bar{r})\,(pa_{i_l}^*).$$

Since $\{pa_{i_l}^*\}$ is bounded and $\{\bar{a}_{i_l}\bar{r}\}$ approaches zero, $\{a_{i_l}' a_{i_l}^*\}$ approaches zero. Thus $\{a_{i_l}'\}$ approaches a^*, $\{\bar{b}_{i_l}\}$ approaches $\bar{l}$ and $\left\{\frac{pa_{i_l} + pb_{i_l}}{a_{i_l} b_{i_l}}\right\}$ approaches 1. This is a contradiction and the theorem is proved

Definition 3. 1. If for any pair of sequences $\{a_i\}$ and $\{b_i\}$ of points approaching $p \in S$ with $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$ the limit.

$$\lim_{i \to \infty} \left\{p + \frac{a_i - b_i}{a_i b_i}\right\} = \bar{c}$$

exists, then the line $\langle p\bar{c}\rangle$ is called the *mixed free tangent* to S at p.

Theorem 3. 6. *In any Banach space a tangent (see definition 2. 6) to S at p is a mixed free tangent.*

Proof. Suppose the tangent at p exists (hence the right hand unit tangent r and the left hand unit tangent l exist and $r = -l$) but $\bar{c}$ (see definition 3. 1) fails to exist. Put again $\bar{r} = p + r$, $\bar{l} = p + l$. Then there exists a pair of sequences $\{a_i\}$ and $\{b_i\}$, $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$ with $pa_i \geqq pb_i$, but $\{\bar{c}_i\} = \left\{p + \frac{a_i - b_i}{a_i b_i}\right\}$ is bounded away from $\bar{r}$.

Now $\{\bar{a}_i\}$ approaches r and $\{\bar{b}_i\}$ approaches $\bar{l}$, hence $\{\bar{c}_i\bar{a}_i\}$ is bounded away from zero. Consider the point x'_i on the line $\langle \bar{c}_i\bar{a}_i \rangle$ such that $x'_i\bar{a}_i = (a'_i\bar{a}_i)(\bar{c}_i\bar{a}_i)$ and and $\bar{a}$ is on the segment $\overline{\bar{c}_i x'_i}$ (see fig. 3). Thus x'_i is on the line $\langle a'_i\bar{b}_i \rangle$. From $p\bar{a}_i = p\bar{c}_i$ it follows that $a'_i\bar{a}_i = a'_i x'_i$, hence $x'_i\bar{b}_i - p\bar{a}_i = a'_i\bar{b}_i - pa'_i$ or

$$x'_i\bar{b}_i = p\bar{a}_i + a'_i\bar{b}_i - pa'_i \leqq p\bar{a}_i + |\, a'_i\bar{b}_i - pa'_i \,| \leqq 1 + 1 = 2.$$

Thus it follows that $\{x'_i\bar{a}_i\}$ is bounded. This means that $\{a'_i\bar{a}_i\} = \left\{\dfrac{x'_i\bar{a}_i}{\bar{c}_i\bar{a}_i}\right\}$ must also be bounded. Since $\{\bar{a}_i\}$ approaches $\bar{r}$ it follows that a subsequence $\{a_i^*\}$ of $\{a_i\}$ exists such that $\{a_i^{*\prime}\}$ converges to a point of the line $\langle p\bar{r} \rangle$ and this contradicts the assumption that $\{\bar{c}_i\}$ is bounded away from $\bar{r}$.

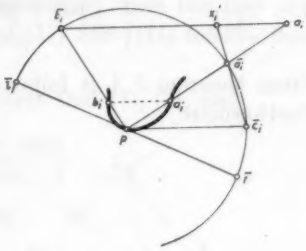


Fig. 3

We state explicitely the definition of the free tangent (see def. 2. 7):

Definition 3. 2. If for any two sequences $\{a_i\}$ and $\{b_i\}$ of points of S, with $a_i \in \overrightarrow{S(b_i)}$ approaching p, the limit $\lim\limits_{i\to\infty}\left\{p + \dfrac{a_i - b_i}{a_i b_i}\right\} = \bar{c}$ exists, the line $\langle p\bar{c} \rangle$ is called the (*unrestricted*) *free tangent* to S at p.

Theorem 3. 7. *A necessary and sufficient condition that S have at p an unrestricted free tangent is that it have at p a right and a left hand free unit tangent, r and l, and that $r = -l$.*

Proof. The *necessity* is immediate from the definition.

Sufficiency: Since right hand and left hand free tangents exist and are negatives of each other the tangent exists and by theorem 3. 6 the mixed free tangent exists. Consider now any two sequences $\{a_i\}$ and $\{b_i\}$ approaching p with $a_i \in \overrightarrow{S(b_i)}$. The sequence

$$\left\{p + \frac{a_i - b_i}{a_i b_i}\right\}$$

can be split into three subsequences, the first of them having a_i and b_i on $\overrightarrow{S(p)}$, the second having a_i and b_i on $\overleftarrow{S(p)}$, and the third having a_i and b_i on opposite sides of p. All of these subsequences have a common limit and it follows easily that the sequence must then have the same limit.

4. Tangents and curvature.

Theorem 4. 1. *A necessary condition that an arc S in a Banach space have a right hand tangent at a point p is that for any two sequences $\{a_i\}$ and $\{b_i\}$ of points of $\overrightarrow{S(p)}$ approaching p with $pa_i \geqq pb_i$, the sequence*

$$\{a_i b_i \cdot K(a_i, b_i, p)\}$$

approach zero.

Proof. We have

$$(a_i b_i)^2 K^2(a_i, b_i, p) =$$

$$\frac{(pa_i + pb_i + a_i b_i)(pa_i + pb_i - a_i b_i)(pa_i - pb_i + a_i b_i)(-pa_i + pb_i + a_i b_i)}{(pa_i)^2 (pb_i)^2},$$

$$(1)\qquad \left\{\frac{pa_i + pb_i + a_ib_i}{pa_i}\right\} = \left\{2 + \frac{pb_i}{pa_i}\left[1 - \frac{pa_i - a_ib_i}{pb_i}\right]\right\}$$

$$(2)\qquad \left\{\frac{pa_i + pb_i - a_ib_i}{pb_i}\right\} = \left\{1 + \frac{pa_i - a_ib_i}{pb_i}\right\};$$

$$(3)\qquad \left\{\frac{pa_i - pb_i + a_ib_i}{pa_i}\right\} = \left\{1 - \frac{pb_i - a_ib_i}{pa_i}\right\},$$

$$(4)\qquad \left\{\frac{-pa_i + pb_i + a_ib_i}{pb_i}\right\} = \left\{1 - \frac{pa_i - a_ib_i}{pb_i}\right\}.$$

From theorem 3. 1 it follows that (1), (2) are bounded and that (4) approaches zero. Furthermore,

$$0 < \frac{pa_i - pb_i + a_ib_i}{pa_i} = 1 - p\hat{b}_i + \bar{a}_i\hat{b}_i \leqq 2$$

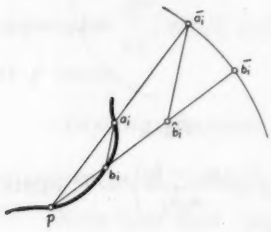


Fig. 4

(see fig. 4). Hence (3) is bounded and

$$\lim_{i\to\infty} \{a_ib_i \cdot K(a_i, b_i, p)\} = 0.$$

Theorem 4. 2. *A necessary and sufficient condition that an arc S in a uniformly convex Banach space have a right hand tangent at a point p is that for any two sequences $\{a_i\}$ and $\{b_i\}$ of points of $\overrightarrow{S(p)}$ approaching p with $pa_i \geqq pb_i$, the sequence*

$$\{a_ib_i \cdot K(a_i, b_i, p)\}$$

approach zero.

Proof. The *necessity* follows from theorem 4. 1.

Sufficiency. Suppose the hypothesis is satisfied but the right hand tangent at p fails to exist. This implies the existence of a sequence of points $\{p_i\}$ of $\overrightarrow{S(p)}$ approaching p such that $\{\bar{p}_i\}$ is not Cauchy. This in turn implies the existence of two subsequences $\{a_i\}$ and $\{b_i\}$ such that $1 \geqq \bar{a}_i\bar{b}_i > \varepsilon_0 > 0$, and $pa_i \geqq pb_i$ for all i and some fixed ε_0. Then evidently

$$\frac{pa_i - pb_i + a_ib_i}{pa_i} = 1 - p\hat{b}_i + \bar{a}_i\hat{b}_i \geqq \bar{a}_i\hat{b}_i > \varepsilon_1 > 0 \text{ for some } \varepsilon_1.$$

From theorem 3. 2 we have that

$$\frac{-pa_i + pb_i + a_ib_i}{pb_i} > \varepsilon_2 > 0 \text{ for a certain } \varepsilon_2.$$

Finally it is at once clear that

$$\frac{pa_i + pb_i + a_ib_i}{pa_i} \geqq 2,$$

$$\frac{pa_i + pb_i - a_ib_i}{pb_i} \geqq 1.$$

Hence $\{a_ib_i \cdot K(a_i, b_i, p)\}$ fails to approach zero and this contradiction shows that the right hand tangent exists.

Theorem 4. 3. *Let S be an arc in a uniformly convex Banach space, p a point of S, and $\{a_i\}$, $\{b_i\}$ any two sequences of points of $\overrightarrow{S(p)}$ approaching p with $pa_i \geqq pb_i$. The following conditions imply each other:*

(I) *S has at p a free right hand tangent.*

(II) *S has at p a right hand tangent and* $\lim_{i\to\infty} \{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0$,

(III) $\lim_{i\to\infty} \{a_ib_i \cdot K(a_i, b_i, p)\} = 0$ *and* $\lim_{i\to\infty} \{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0$.

(IV) $$\lim_{i\to\infty} \sqrt{(\{a_ib_i)^2 + pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0.$$

Proof. The equivalence of (III) and (IV) is seen immediately from the fact that distances are nonnegative real numbers. The equivalence of (II) and (III) follows from theorem 4. 2.

(I) *implies* (II): From theorems 3. 1 and 3. 3 we conclude that (1), (2), and

$$\left\{\frac{pa_i - pb_i + a_ib_i}{a_ib_i}\right\} = \left\{1 + \frac{pa_i - pb_i}{a_ib_i}\right\} \tag{5}$$

are bounded and that

$$\left\{\frac{-pa_i + pb_i + a_ib_i}{a_ib_i}\right\} = \left\{1 - \frac{pa_i - pb_i}{a_ib_i}\right\} \tag{6}$$

approaches zero.

Hence

$$\lim_{i\to\infty} \{pa_i \cdot pb_i \cdot K^2(a_i, b_i, p)\} =$$

$$\lim_{i\to\infty} \left\{\frac{(pa_i + pb_i + a_ib_i)\,(pa_i + pb_i - a_ib_i)\,(pa_i - pb_i + a_ib_i)\,(-pa_i + pb_i + a_ib_i)}{pa_i \cdot pb_i \cdot (a_ib_i)^2}\right\} = 0.$$

(II) *implies* (I): From theorem 3. 1 it follows that (1) and (2) do not approach zero. Since $pa_i \geqq pb_i$, (5) is bounded away from zero. Thus if $\{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\}$ is to approach zero it must be that (6) approaches zero or that $\left\{\frac{pa_i - pb_i}{a_ib_i}\right\}$ approaches one. But theorem 3. 4 then guarantees the existence of a right hand free tangent at p.

Definition 4. 1. If the right and left hand unit tangents of an arc S in a Banach space exist at a point $p \in S$ and are equal, we say that S has a *cusp* at p.

Theorem 4. 4. *A necessary and sufficient condition that an arc S in a uniformly convex Banach space have at p either a cusp or a tangent is that for any two sequences* $\{a_i\}$ *and* $\{b_i\}$ *of points of S approaching p, the sequence*

$$\{a_ib_i \cdot K(a_i, b_i, p)\}$$

approach zero.

Proof. Define $f(a_i, b_i) \equiv a_ib_i \cdot K(a_i, b_i, p)$ (f depending on the parameter p). Clearly $f(a_i, b_i) = f(b_i, a_i)$, hence $pa_i \geqq pb_i$ can be assumed for all i.

Necessity: Let $\{a_i, b_i\}$ be any sequence of pairs $a_i, b_i \in S$; $(p \neq a_i, b_i)$, such that $pa_i \geqq pb_i$. This sequence can be split into four subsequences:

$$\{a_i^{(1)}, b_i^{(1)}\} \text{ such that } a_i^{(1)} \in \overrightarrow{S(p)},\ b_i^{(1)} \in \overrightarrow{S(p)},$$

$$\{a_i^{(2)}, b_i^{(2)}\} \text{ such that } a_i^{(2)} \in \overleftarrow{S(p)},\ b_i^{(2)} \in \overleftarrow{S(p)},$$

$$\{a_i^{(3)}, b_i^{(3)}\} \text{ such that } a_i^{(3)} \in \overrightarrow{S(p)},\ b_i^{(3)} \in \overleftarrow{S(p)},$$

$$\{a_i^{(4)}, b_i^{(4)}\} \text{ such that } a_i^{(4)} \in \overleftarrow{S(p)},\ b_i^{(4)} \in \overrightarrow{S(p)}.$$

Any element belongs to exactly one of these subsequences. The definitions of the cusp and the tangent guarantee the existence of the left and right hand tangents. From theorem 4. 2 it follows that $\{f(a_i^{(1)}, b_i^{(1)})\}$ and (by interchanging "left" and "right") that $\{f(a_i^{(2)}, b_i^{(2)})\}$ approach zero.

If S has a cusp at p, the proof of theorem 4. 2 can be applied without any change to the sequences $\{a_i^{(3)}\}$ and $\{b_i^{(3)}\}$. Hence in this case $\{f(a_i^{(3)}, b_i^{(3)})\}$ has zero as limit, and this is also true for $\{f(a_i^{(4)}, b_i^{(4)})\}$.

Now we assume that S has a tangent at p. From theorem 3. 5 it follows that

$$\left\{\frac{pa_i^{(3)} + pb_i^{(3)}}{a_i^{(3)} b_i^{(3)}}\right\}$$

approaches one. $\bar{a}_i^{(3)} \hat{b}_i^{(3)}$ is bounded. Thus

$$\left\{\frac{pa_i^{(3)} + pb_i^{(3)} + a_i^{(3)} b_i^{(3)}}{pa_i^{(3)}}\right\} = \left\{\bar{a}_i^{(3)} \hat{b}_i^{(3)} \left[\frac{pa_i^{(3)} + pb_i^{(3)}}{a_i^{(3)} b_i^{(3)}} + 1\right]\right\}$$

is bounded.

By arguments similar to those in the proof of theorem 3. 1 (b_i and p interchanged) it follows that

$$\left\{\frac{pa_i^{(3)} + pb_i^{(3)} - a_i^{(3)} b_i^{(3)}}{pb_i^{(3)}}\right\}$$

approaches zero. Furthermore,

$$\left\{\frac{pa_i^{(3)} - pb_i^{(3)} + a_i^{(3)} b_i^{(3)}}{pa_i^{(3)}}\right\} = \{1 - p\hat{b}_i^{(3)} + \bar{a}_i^{(3)} \hat{b}_i^{(3)}\}$$

is bounded. Finally, for large i, we may assume that $a_i^{(3)} b_i^{(3)} \geqq pa_i^{(3)}$, hence

$$0 < \frac{-pa_i^{(3)} + pb_i^{(3)} + a_i^{(3)} b_i^{(3)}}{pb_i^{(3)}} = -pa_i^{(3)\prime} + p\bar{b}_i^{(3)} + a_i^{(3)\prime} \bar{b}_i^{(3)} \leqq p\bar{b}_1^{(3)} = 1,$$

and the sequence (4) is bounded. Now $\{f(a_i^{(3)}, b_i^{(3)})\}$ is seen to approach zero.

By interchanging "left" and "right" we conclude that $\{f(a_i^{(4)}, b_i^{(4)})\}$ approaches zero, too. Thus $\{f(a_i, b_i)\}$ has zero as limit.

Sufficiency: From theorem 4. 2 it follows that the right hand unit tangent r and the left hand unit tangent l exist. It remains to show that either $r = -l$ or $r = l$.

Consider two sequences $\{a_i\}$, $\{b_i\}$ such that $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$, and $pa_i = pb_i$, for all i. Then $\{\bar{a}_i\}$ approaches $\bar{r}$, $\{\bar{b}_i\}$ approaches $\bar{l}$. Since

$$\lim_{i \to \infty} [f(a_i, b_i)]^2 = \lim_{i \to \infty} \frac{2pa_i + a_i b_i}{pa_i} \frac{2pa_i - a_i b_i}{pa_i} \left(\frac{a_i b_i}{pa_i}\right)^2 = 0$$

it follows that either

$$\lim_{i \to \infty} (2 - \bar{a}_i \bar{b}_i) = 2 - \bar{r}\bar{l} = 0,$$

hence $r = -l$, or that

$$\lim_{i \to \infty} \bar{a}_i \hat{b}_i = \lim_{i \to \infty} \bar{a}_i \hat{b}_i = 0,$$

hence $r = l$. In the first case, S has a tangent at p, in the latter case, S has a cusp at p.

Theorem 4. 5. *Let S be an arc in a uniformly convex Banach space, p a point of S, and $\{a_i\}$, $\{b_i\}$ any two sequences of points of S approaching p. The following conditions are equivalent:*

(I) *S has at p an unrestricted free tangent.*

(II) *S has at p a tangent and* $\lim_{i\to\infty} \{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0$.

(III) $\lim_{i\to\infty} \{a_ib_i \cdot K(a_i, b_i, p)\} = 0$ *and* $\lim_{i\to\infty} \{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0$.

(IV) $\lim_{i\to\infty} \{\sqrt{(a_ib_i)^2 + pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0$.

Proof. The equivalence of (II), (III), and (IV) is seen immediately. (For proving (II) replace $f(a_i, b_i)$ in the sufficience proof of theorem 4. 4 by 2nd therm of (III)).

(I) *implies* (II): Evidently it is sufficient to show that for $a_i \in \overrightarrow{S(p)}$, $b_i \in \overleftarrow{S(p)}$, $pa_i \geqq pb_i$ we have

$$\lim_{i\to\infty} \{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\} = 0.$$

For large i, it follows from the existence of the free tangent that $pa_i \leqq a_ib_i$ and $pb_i \leqq a_ib_i$. Hence (5) and (6) are bounded. As in the proof of theorem 4. 4 we conclude that (1) is bounded, too, and that (2) approaches zero. Thus $\{\sqrt{pa_i \cdot pb_i} \cdot K(a_i, b_i, p)\}$ has zero as limit.

(II) *implies* (I): From theorem 4. 3 it follows that the free right hand unit tangent r and the free left hand unit tangent l exist, and from the existence of the tangent we have $r = -l$. Hence we conclude from theorem 3. 7 that S has an unrestricted free tangent at p.

Corollary 4. 5. 1. *A sufficient condition that an arc S in a uniformly convex Banach space have at p an unrestricted free tangent is that Alt curvature exist at p.*

Theorem 4. 6. *A necessary and sufficient condition that an arc S in a uniformly convex Banach space have at p an unrestricted free tangent is that for any three sequences* $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ *of S approaching p, the sequence*

$$\{\sqrt{a_ib_i \cdot b_ic_i + b_ic_i \cdot c_ia_i + c_ia_i \cdot a_ib_i} \cdot K(a_i, b_i, c_i)\}$$

approach zero.

Proof. Define $g(a_i, b_i, c_i) \equiv \sqrt{a_ib_i \cdot b_ic_i + b_ic_i \cdot c_ia_i + c_ia_i \cdot a_ib_i} \cdot K(a_i, b_i, c_i)$. Since $g(a_i, b_i, c_i) = g(b_i, a_i, c_i) = g(c_i, a_i, b_i)$ etc. we may assume that $c_ib_i \leqq c_ia_i$ and $b_ia_i \leqq c_ia_i$.

Necessity: If we replace, in the proofs of theorems 3. 1 and 3. 3, the point p formally by c_i, we see immediately that all conclusions remain valid. Hence it follows as in the proof of theorem 4. 3 that

$$\{c_ia_i \cdot c_ib_i \cdot K^2(a_i, b_i, c_i)\} \tag{7}$$

approaches zero. The same is true if p is replaced by a_i, hence

$$\{a_ib_i \cdot a_ic_i \cdot K^2(a_i, b_i, c_i)\} \tag{8}$$

approaches zero. Finally, from $a_ib_i \cdot b_ic_i \leqq a_ib_i \cdot a_ic_i$ it follows that

$$\{a_ib_i \cdot b_ic_i \cdot K^2(a_i, b_i, c_i)\} \tag{9}$$

has zero as limit. Thus by adding and drawing the square root we have

$$\lim_{i\to\infty} \{g(a_i, b_i, c_i)\} = 0.$$

Sufficiency: From $\lim_{i\to\infty} \{g(a_i, b_i, c_i)\} = 0$ we conclude that the sequences (7), (8), (9) approach zero. Hence by adding we obtain that

$$\lim_{i\to\infty} \{a_ib_i(a_ic_i + b_ic_i)\, K^2(a_i, b_i, c_i)\} \geqq \lim_{i\to\infty} \{(a_ib_i)^2 K^2(a_i, b_i, c_i)\} = 0.$$

By the special choice $c_i = p$, for all i, the existence of the unrestricted free tangent follows from theorem 4. 5.

Corollary 4. 6. 1. *Corollary 4. 5. 1 holds if Alt curvature is replaced by Menger curvature.*

Theorem 4. 7. *A necessary and sufficient condition that a rectifiable arc S in a uniformly convex Banach space have at p an unrestricted free tangent is that for any two sequences $\{a_i\}$ and $\{b_i\}$ of S approaching p, the sequence*

$$\{a_i b_i \cdot K_H(a_i, b_i)\}$$

approach zero.

Proof. By definition we have

$$K_H^2(a_i, b_i) = 4!\ \frac{l(a_i, b_i) - a_i b_i}{l^3(a_i, b_i)},$$

where $l(a_i, b_i)$ denotes the arc length of the subarc of S with endpoints a_i and b_i. We put

$$l(a_i, b_i) = l, \quad a_i b_i = d.$$

Then

$$(a_i b_i)^2 \cdot K_H^2(a_i, b_i) = d^2 \cdot 4!\ \frac{l - d}{l^3} = 4!\ \frac{\frac{l}{d} - 1}{\left(\frac{l}{d}\right)^3}.$$

Necessity: If the free tangent exists then evidently[8]) $\left\{\frac{l}{d}\right\}$ approaches one, hence

$$\{a_i b_i \cdot K_H(a_i, b_i)\}$$

approaches zero.

Sufficiency: Suppose that $\{a_i b_i \cdot K_H(a_i, b_i)\}$ approaches zero but that the free tangent fails to exist. This implies that there are three sequences $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ of points of S such that the lines $\langle a_i c_i \rangle$ and $\langle b_i c_i \rangle$ do not approach each other. This can be stated in the following way: Define

$$\bar{a}_i = c_i + \frac{a_i - c_i}{a_i c_i}$$

$$\bar{b}_i = c_i + \frac{b_i - c_i}{b_i c_i}$$

$$a_i' = c_i + \frac{a_i - c_i}{b_i c_i} \qquad \text{(see fig. 5).}$$

Fig. 5

We may assume that $a_i c_i \geqq b_i c_i$ and $a_i c_i \geqq a_i b_i$. Then our assumption is that $\{\bar{a}_i \bar{b}_i\}$ is bounded away from zero and from two.

Denote by x_i' the point on the segment $a_i' \bar{b}_i$ such that $a_i' \bar{a}_i = a_i' x_i'$. Since we assume that

$$\left\{\frac{\frac{l}{d} - 1}{\left(\frac{l}{d}\right)^3}\right\}$$

approaches zero it follows that either $\left\{\frac{l}{d}\right\}$ approaches one or that $\left\{\frac{d}{l}\right\}$ approaches zero.

---

[8]) $\frac{l}{d} = l^*$ can be interpreted as length of an arc S^* approaching pointwise a line segment d^* of length one. If l^* does not approach one, infinitely many pairs p_i, q_i can be found on S^* such that the lines $\langle p_i q_i \rangle$ do not approach the tangent at p (supported by d^*). This contradicts the existence of the free tangent at p.

Suppose $\left\{\frac{d}{l}\right\}$ approaches zero. Since S is rectifiable we can find points e_i, f_i on each subarc of S with endpoints a_i, b_i such that

$$\frac{l(e_i, f_i)}{e_i f_i} = 2, \text{ for large } i, \text{ and } \{e_i\}, \{f_i\} \text{ approach } p.$$

But then

$$\{e_i f_i \cdot K_H(e_i, f_i)\}$$

would fail to approach zero. Thus we conclude that $\left\{\frac{l}{d}\right\}$, $\left\{\frac{l(b_i, c_i)}{b_i c_i}\right\}$, and $\left\{\frac{l(a_i, c_i)}{a_i c_i}\right\}$ approach one.

The points $c_i + \frac{x - c_i}{b_i c_i}$ with fixed b_i, c_i and variable $x \in S$ form an arc S' (see fig. 5). a_i', $\bar{b}_i$, c_i are points of S'. If we denote by l' the length of the subarc of S' with endpoints a_i', $\bar{b}_i$, by l'' that one with endpoints $\bar{b}_i$, c_i, and by l'' that one with endpoints a_i', c_i, we obtain that

$$\frac{l}{d} = \frac{l'}{a_i' \bar{b}_i},$$

$$\frac{l(b_i, c_i)}{b_i c_i} = \frac{l''}{\bar{b}_i c_i},$$

$$\frac{l(a_i, c_i)}{a_i c_i} = \frac{l'''}{a_i' c_i} \qquad \text{(see fig. 5).}$$

It now follows that $\{l' - a_i'\bar{b}_i\}$, $\{l'' - \bar{b}_i c_i\}$, and $\{l''' - a_i' c_i\}$ approach zero. If $l''' = l' + l''$ we conclude that

$$\{a_i'\bar{b}_i + \bar{b}_i c_i - a_i' c_i\} = \{\bar{b}_i x_i'\}$$

approaches zero. But since $\{\bar{a}_i \bar{b}_i\}$ is bounded away from zero, this contradicts uniform convexity. If $l'' = l' + l'''$ it follows from $a_i'\bar{b}_i + c_i a_i' - c_i \bar{b}_i \geqq \bar{a}_i \bar{b}_i$ that $\bar{a}_i \bar{b}_i$ approaches zero, and in case $l' = l'' + l'''$ the inequality $a_i' c_i + \bar{b}_i c_i - a_i' \bar{b}_i \geqq 1$ leads to a contradiction.

Thus therorem 4. 7 is proved.

Corollary 4. 7. 1. *Corollary* 4. 5. 1 *holds if Alt curvature is replaced by Haantjes curvature.*

References.

[1] *L. M. Blumenthal*, Distance Geometry, Oxford Press, 1953.

[2] *L. M. Blumenthal*, Note on an Arc without Tangents, Reports of a Mathematics Colloquium, Publications of the University of Notre Dame, Second Series, **4** (1942), 1—3.

[3] *H. Busemann*, The Foundations of Minkowskian Geometry, Comm. Math. Helv. **24** (1950), 156—187.

[4] *J. A. Clarkson*, Uniformly Convex Spaces, Trans. Amer. Math. Soc. **40** (1936), 396—414.

[5] *E. Egerváry* and *G. Alexits*, Fondaments d'une Théorie Générale de la Courbure Linéaire, Comm. Math. Helv. **13** (1941), 257—276.

[6] *A. R. Lovaglia*, Locally uniformly convex Banach Spaces, Trans. Amer. Math. Soc. **78** (1955), 225—238.

[7] *H. Rund*, The Differential Geometry of Finsler Spaces, Berlin-Heidelberg 1959.

[8] *I. J. Schoenberg*, A Remark on M. M. Day's Characterization of inner product Spaces and a Conjecture of L. M. Blumenthal. Proc. Amer. Math. Soc. **3** (1952), 961—964.

Eingegangen 31. Juli 1959.

Arithmetic on curves of genus 1.

II. A general result.

By *J. W. S. Cassels* in Cambridge (England).

Contents.

§ 1. Introduction.
§ 2. Preliminaries.
§ 3. Extensions of coverings.
§ 4. A mutual invariant of two invariance classes.
§ 5. The local reciprocity.
§ 6. Local and global extensibility.
§ 7. The global theory.
Appendix.
References.

§ 1. Introduction.

Let $\mathcal{C}$ be a curve of genus 1 defined over an algebraic number-field $\mathfrak{K}$ and let $\mathfrak{o}$ be a point of $\mathcal{C}$ defined over $\mathfrak{K}$. The points of $\mathcal{C}$ defined over $\mathfrak{K}$ may be given the structure of an abelian group if

$$\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$$

is defined to mean that the divisor[1]) $(\mathfrak{a})(\mathfrak{b})$ is linearly equivalent on $\mathcal{C}$ to the divisor $(\mathfrak{c})(\mathfrak{o})$. A well-known theorem of Mordell and Weil (see Weil (**13**)) states that the group $\mathfrak{G}$ of points of $\mathcal{C}$ defined over $\mathfrak{K}$ is finitely generated.

Now let

$$\mathfrak{x} \to \nu\mathfrak{x} \tag{1}$$

be a mapping defined over $\mathfrak{K}$ of the curve $\mathcal{C}$ into itself which takes $\mathfrak{o}$ into $\mathfrak{o}$. Here ν can either be an ordinary integer or (1) can be "complex multiplication" by an integer of a complex quadratic field. It follows from the theorem of Mordell-Weil that the group $\mathfrak{G}/\nu\mathfrak{G}$, where $\nu\mathfrak{G}$ is the image of $\mathfrak{G}$ under (1), is finitely generated. In fact, however, the proof of the finiteness of $\mathfrak{G}/\nu\mathfrak{G}$, at least for some ν, is an essential step in the proof of the full theorem. Before enunciating the principle result of this paper, it will be necessary to discuss the proof of the finiteness of $\mathfrak{G}/\nu\mathfrak{G}$ and it is convenient to adopt a rather different point of view from the original[2]).

---

[1]) Divisors on C will be written multiplicatively.

[2]) The present point of view is more closely related to that of Roquette (7). The notion of covering is suggested by the work of Lang and Serre (cf. Serre (**11**)) but the present treatment is less sophisticated since we deal throughout with fields of characteristic 0.

We say that there is a ν-covering U of $\mathcal{C}$ if there is a curve $\mathcal{D}$, defined over $\mathfrak{K}$, and a commutative triangle

$$(2) \qquad U: \quad \begin{tikzcd} C \arrow[r] & C \\ \mathcal{D} \arrow[u, leftrightarrow] \arrow[ur] & \end{tikzcd}$$

with associated generic points

$$(3) \qquad \begin{tikzcd} \mathfrak{x}_1 \arrow[r] & \mathfrak{x} = \nu\mathfrak{x}_1, \\ \mathfrak{X} \arrow[u, leftrightarrow] \arrow[ur] & \end{tikzcd}$$

where the rational map $\mathfrak{X} \to \mathfrak{x}$ is defined over $\mathfrak{K}$ and the birational correspondence $\mathfrak{x}_1 \longleftrightarrow \mathfrak{X}$ is defined over the algebraic closure $\bar{\mathfrak{K}}$ of $\mathfrak{K}$. We shall say that the curve $\mathcal{D}$ and the set of mappings (2) gives the same ν-covering as a curve $\mathcal{D}$ and a similar set (2′) of mappings if there is a birational correspondence $\mathfrak{X} \longleftrightarrow \mathfrak{X}'$ defined over $\mathfrak{K}$ between the generic points of $\mathcal{D}$ and $\mathcal{D}'$ and a birational correspondence $\mathfrak{x}_1' = \mathfrak{x}_1 + \mathfrak{d}$, where $\mathfrak{d}$ is any solution of $\nu\mathfrak{d} = \mathfrak{o}$, defined over $\bar{\mathfrak{K}}$, such that the following diagram is commutative:

$$(4) \qquad \begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x}_1 \arrow[d, leftrightarrow] \\ \mathfrak{X}' \arrow[r] & \mathfrak{x}_1' = \mathfrak{x}_1 + \mathfrak{d}. \end{tikzcd}$$

The existence of a diagram (4) clearly does define an equivalence relation between diagrams (2), so the notation of ν-covering is well-defined.

We shall write $\mathcal{D} = \mathcal{D}(U)$ to mean that $\mathcal{D}$ is the curve in a representation (2) of U: so $\mathcal{D}(U)$ is determined up to a birational transformation defined over $\mathfrak{K}$.

If $\mathfrak{a}$ is any point on $\mathcal{C}$, there is a covering U in which the inverse image of $\mathfrak{a}$ under the map $\mathfrak{X} \to \mathfrak{x}$ is a point $\mathfrak{A}$ defined over $\mathfrak{K}$. Indeed if $\mathfrak{a}_1$ is any solution of $\nu\mathfrak{a}_1 = \mathfrak{a}$, we can take $\mathcal{D} = \mathcal{D}(U) = \mathcal{C}$, $\mathfrak{A} = \mathfrak{o}$, and the mapping $\mathfrak{x}_1 \longleftrightarrow \mathfrak{X}$ to be $\mathfrak{X} = \mathfrak{x}_1 - \mathfrak{a}_1$. It is not difficult to see that, in fact, the covering, in the sense just defined, is uniquely defined by $\mathfrak{a}$: for if $\mathfrak{A}$ is any point of $\mathcal{D}$ defined over $\mathfrak{K}$ there is a birational correspondence

$$\mathfrak{x} \longleftrightarrow \mathfrak{A} + \mathfrak{x}$$

of $\mathcal{C}$ with $\mathcal{D}$ defined over $\mathfrak{K}$, since $\mathcal{C}$ is the Jacobian of $\mathcal{D}$ (cf. § 2 below).

The work of Weil (14) sets up a $1-1$ correspondence between the elements of $\mathfrak{G}/\nu\mathfrak{G}$ and the ν-coverings U for which there is a point $\mathfrak{A}$ defined over $\mathfrak{K}$ on the curve $\mathcal{D}(U)$. There is no known way of finding effectively and infallibly in every case whether there is such a point on $\mathcal{D}(U)$. It is, however, possible to determine whether there is a point $\mathfrak{A}_\mathfrak{p}$ on $\mathcal{D}(U)$ defined over the closure $\mathfrak{K}_\mathfrak{p}$ of $\mathfrak{K}$ with respect to each valuation $\mathfrak{p}$. It turns out, as Weil showed[3]), that there are only a finite number of ν-coverings U for which $\mathcal{D}(U)$ has a point everywhere locally. We shall say that such ν-coverings "survive

[3]) Other proofs in Roquette (7) and Lang-Tate (6).

the first descent". There is a natural structure of an abelian group on the ν-coverings[4]) of $\mathcal{C}$ and under this the coverings which survive the first descent form a group $M^{(1)}$.

One may similarly use ν^2-coverings instead of ν-coverings. Our object is to set up a relationship between these two types of covering. We shall say that a ν^2-covering

$$(5) \qquad \mathcal{V}: \quad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{E} \arrow[ur] & \end{tikzcd}$$

extends the ν-covering U given by (2) if there is a commutative diagram

$$(6) \qquad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{E} \arrow[r] & \mathcal{D} \arrow[ur] & \end{tikzcd}$$

where the birational maps $\mathcal{C} \longleftrightarrow \mathcal{E}$, $\mathcal{C} \longleftrightarrow \mathcal{D}$ are defined over[5]) $\bar{\mathfrak{K}}$ but the remaining maps are defined over $\mathfrak{K}$. Let corresponding generic points be as in the following diagram

$$(7) \qquad \begin{tikzcd} \mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x}_1 = \nu\mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x} = \nu\mathfrak{x}_1. \\ \mathbf{X} \arrow[r] & \mathfrak{X} \arrow[ur] & \end{tikzcd}$$

If there is a point $\mathfrak{A}$ on $\mathcal{D}(U)$ defined over $\mathfrak{K}$, then it is certainly possible to extend the ν-covering U to a ν^2-covering $\mathcal{V}$ and indeed with $\mathcal{E}(\mathcal{V}) = \mathcal{C}$. For let $\mathfrak{A}$ map onto $\mathfrak{a}_1$ under $\mathfrak{X} \longleftrightarrow \mathfrak{x}_1$ and let $\mathfrak{a}_2$ be any solution of $\nu\mathfrak{a}_2 = \mathfrak{a}_1$. Then we can take the map $\mathfrak{x}_2 \longleftrightarrow \mathbf{X}$ as $\mathbf{X} = \mathfrak{x}_2 - \mathfrak{a}_2$ and the map $\mathbf{X} \longleftrightarrow \mathfrak{X}$ as $\mathfrak{X} = \mathfrak{A} + \nu\mathbf{X}$.

If $\mathfrak{K}$ is an algebraic number-field, then the existence of a point on $\mathcal{D}(U)$ over each local closure $\mathfrak{K}_\mathfrak{p}$ of $\mathfrak{K}$ does not ensure that there is a point on $\mathcal{D}(U)$ defined over $\mathfrak{K}$, as Selmer (8) has shown. On the other hand, we shall prove the following result[6]) under certain restrictions which are probably unnecessary: suppose that a ν-covering U is given defined over $\mathfrak{K}$, and that for each local closure $\mathfrak{K}_\mathfrak{p}$ there is a ν^2-covering $\mathcal{V}_\mathfrak{p}$ defined over $\mathfrak{K}$ which extends it: then there exists an extension $\mathcal{V}$ of U defined over $\mathfrak{K}$. The extension $\mathcal{V}$ of U is not, of course, uniquely determined by U.

We say that a ν-covering U of $\mathcal{C}$ "survives the second descent" if there is a ν^2-extension $\mathcal{V}$ of U for which there is everywhere locally a point $\mathbf{A}_\mathfrak{p}$ on the curve $\mathcal{E}(\mathcal{V})$. The ν-coverings with this property can always be effectively determined for any given

---

[4]) This group is related to, but not identical with, the group of homogeneous spaces of Weil (14). It appears, of course, in more or less recognizable shape in much earlier work on special curves. For the special curve $x^4 + by^2 + c = z^2$ and $\nu = 2$ it occurs in Reichardt (15) practically from the point of view adopted here. I am grateful to Professor Hasse for drawing my attention to this interesting paper which appears to have been overlooked by later workers in this field, and which is also notable for a novel type of infinite descent. For other references to earlier works see the bibliographies of (2) and (5).

[5]) $\bar{\mathfrak{K}}$ is the algebraic closure of $\mathfrak{K}$.

[6]) Precise enunciation in § 6, theorem IV.

$\mathfrak{K}$ and $\mathcal{C}$. The ν-coverings U which survive the second descent form a subgroup $M^{(2)}$ of the group $M^{(1)}$. If we denote[7]) by M the group of ν-coverings for which there is actually a point on $\mathcal{D}(U)$ defined over $\mathfrak{K}$, then clearly

$$M < M^{(2)} < M^{(1)}.$$

Hence $M^{(2)}$ is, in general, better than $M^{(1)}$ in estimating the number of generators of $\mathfrak{G}$.

In this note I shall prove the following theorem:

Theorem I. Suppose that

(I) *ν^2 is a rational integer.*

(II) *Every point of the kernel of the mapping $\mathfrak{x}_1 \to \nu\mathfrak{x}_1$ is defined over $\mathfrak{K}$.*

Then there is a mapping $T(U', U'')$ of pairs U', U'' of elements of $M^{(1)}$ into the rationals modulo 1, which is a skew-symmetric bilinear form in the sense that

$$T(U', U'') + T(U'', U') = 0$$

and

$$T(U'U''', U'') = T(U', U'') + T(U''', U'').$$

The elements U of $M^{(2)}$ are precisely the kernel of T, that is they are the $U \in M^{(1)}$ such that $T(U, U') = 0$ for all $U' \in M^{(1)}$.

In the first paper of this series (Cassels (2)) I proved this result by special arguments in the very special case that $\mathcal{C}$ is a curve

$$x^3 + y^3 + dz^3 = 0$$

and $\nu = \sqrt{-3}$. I suspect[8]) that the condition (II) in the enunciation of the theorem is only necessitated by deficient techniques of proof. On the other hand, (I) seems to be a very natural restriction[9]).

The theorem and its proof shows that to determine the coverings of the second descent it is not necessary to work[10]) in a field of definition of the points of the kernel of the mapping $\mathfrak{x}_2 \to \nu^2\mathfrak{x}_2$ but only in the corresponding field for the map $\mathfrak{x}_1 \to \nu\mathfrak{x}_1$. If one defines third, fourth, . . . descents in the obvious way, it is natural to suppose that these also can be treated without extending the field of definition. I hope to come back to this point, as also to the question whether condition (II) is necessary, on a later occasion.

As the reader will easily convince himself, some of the lemmas are true without the assumption of conditions (I), (II) of Theorem I. (All of § 2 except the explicit determination of the group of invariance classes, for example). However a more elaborate notation would be required to give all results in full generality and, since conditions (I), (II) are vitally needed at several stages in the argument, we have assumed them from the very beginning.

§ 2. Preliminaries.

In this section we set the stage and introduce the principal characters. All the results must be well-known, at least in principle, but we give some proofs in order to introduce arguments which we shall need later.

---

[7]) So M is isomorphic to $\mathfrak{G}/\nu\mathfrak{G}$ by Weil's theorems.

[8]) Selmer (10) has given some evidence that the phenomenon which he discovered and which originated this investigation continues to appear when (II) is not satisfied. See also the comparison with the results of Tate (12) in § 5 after the enunciation of theorem III.

[9]) What we actually use is that the map $\mathfrak{x}_1 \to \nu\mathfrak{x}_1$ and $\mathfrak{x}_1 \to \bar{\nu}\mathfrak{x}_1$ have the same kernel, where $\bar{\nu}$ is a conjugate of ν.

[10]) For numerical examples, compare the first paper of this series (Cassels (2)).

In § 2, the field $\mathfrak{K}$ is any field of characteristic 0. For simplicity we make the assumptions (I) (II) of the theorem. We shall denote by $\mathfrak{d}$ any solution of

$$\nu\mathfrak{d} = \mathfrak{o}. \tag{1}$$

Under condition (I) of the theorem, it is easy to see that the inverse image of the divisor[11] $(\mathfrak{o})^{-1}(\mathfrak{d})$ under the map $\mathfrak{x} \to \nu\mathfrak{x}$ is linearly equivalent to 0. Since $\mathfrak{d}$ is defined over $\mathfrak{K}$ by condition (II) of the theorem, there is thus a function $\Phi_{\mathfrak{d}}(\mathfrak{x})$ defined over $\mathfrak{K}$ with this divisor. If d is the order of $\mathfrak{d}$, so that the divisor $\{(\mathfrak{o})^{-1}(\mathfrak{d})\}^d$ is linearly equivalent to 0, and if $\mathfrak{x} = \nu\mathfrak{x}_1$, where $\mathfrak{x}_1$ is a generic point of C, it is clear that $\{\Phi_{\mathfrak{d}}(\mathfrak{x}_1)\}^d \in \mathfrak{K}(\mathfrak{x})$ is a function of $\mathfrak{x}$ with divisor $\{(\mathfrak{o})^{-1}(\mathfrak{d})\}^d$. Hence, if $\mathfrak{e}$ is any other solution of

$$\nu\mathfrak{e} = \mathfrak{o} \tag{2}$$

we have

$$\Phi_d(\mathfrak{x}_1 + \mathfrak{e}) = \chi(\mathfrak{d}, \mathfrak{e})\, \Phi_d(\mathfrak{x}_1), \tag{3}$$

where $\chi(\mathfrak{d}, \mathfrak{e})$ is a d-th root of 1.

It is well-known and easy to prove that $\chi(\mathfrak{d}, \mathfrak{e})$ is a multiplicative bilinear form, that is, that

$$\chi(\mathfrak{d}_1 + \mathfrak{d}_2, \mathfrak{e}) = \chi(\mathfrak{d}_1, \mathfrak{e})\, \chi(\mathfrak{d}_2, \mathfrak{e}), \tag{4}$$

$$\chi(\mathfrak{d}, \mathfrak{e}_1 + \mathfrak{e}_2) = \chi(\mathfrak{d}, \mathfrak{e}_1)\, \chi(\mathfrak{d}, \mathfrak{e}_2). \tag{5}$$

If $\nu = n$ is a rational integer the bilinear form has the special property

$$\chi(\mathfrak{d}, \mathfrak{d}) = 1 \qquad (\text{if } \nu = n). \tag{6}$$

Further, in any case, $\chi(\mathfrak{d}, \mathfrak{e})$ is non-degenerate in the sense that if $\chi(\mathfrak{d}, \mathfrak{e}) = 1$ for all $\mathfrak{e}$ and some particular $\mathfrak{d}$, then $\mathfrak{d} = \mathfrak{o}$.

Since the functions $\Phi_{\mathfrak{d}}(\mathfrak{x})$ and the points $\mathfrak{d}$, $\mathfrak{e}$ are defined over $\mathfrak{K}$ by hypothesis (II), it follows from (3) that

$$\chi(\mathfrak{d}, \mathfrak{e}) \in \mathfrak{K}. \tag{7}$$

The relation (3), together with non-degeneracy of $\chi(\mathfrak{d}, \mathfrak{e})$ shows that $\mathfrak{x}_1$ is completely specified by $\mathfrak{x} = \nu\mathfrak{x}_1$ and the $\Phi_{\mathfrak{d}}(\mathfrak{x}_1)$.

To avoid the separate discussion of special cases we use

Lemma 1. *Let $\mathfrak{a}_1, \ldots, \mathfrak{a}_s$ be any number of points on C defined over the algebraic closure of $\mathfrak{K}$. Then there is a function $g(\mathfrak{x})$ of the point $\mathfrak{x}$ on C which is defined over $\mathfrak{K}$ and such that*

$$\varphi_{\mathfrak{d}}(\mathfrak{x}_1) = \frac{\Phi_{\mathfrak{d}}(\mathfrak{x}_1)\, g(\nu\mathfrak{x}_1)}{g(\nu\mathfrak{x}_1 - \mathfrak{d})} \tag{8}$$

has neither a pole nor a zero at any of $\mathfrak{x}_1 = \mathfrak{a}_1, \ldots, \mathfrak{a}_s$.

For it is only necessary to choose $g(\mathfrak{x})$ to have a simple zero at $\mathfrak{x} = \mathfrak{o}$ and to be neither 0 nor ∞ at $\mathfrak{x} = \nu\mathfrak{a}_j + \mathfrak{d}$ $(1 \leqq j \leqq s,\ \nu\mathfrak{d} = \mathfrak{o})$ when these points differ from $\mathfrak{o}$.

We shall often use the following result without special reference.

Lemma 2. *Let $\mathfrak{x}_1$ be a generic point on C. If $\mathfrak{d}_1, \ldots, \mathfrak{d}_s$ are solutions of $\nu\mathfrak{d}_j = 0$ $(1 \leqq j \leqq s)$ and if $m_1, \ldots, m_s$ are integers such that*

$$m_1\mathfrak{d}_1 + \cdots + m_s\mathfrak{d}_s = \mathfrak{o}, \tag{9}$$

[11]) We write divisors multiplicatively. The divisor of a function is (divisor of zeros) · (divisor of poles)$^{-1}$.

then

$$\{\varphi_{\mathfrak{d}_1}(\mathfrak{x}_1)\}^{m_1} \cdots \{\varphi_{\mathfrak{d}_s}(\mathfrak{x}_1)\}^{m_s} \in \mathfrak{K}(\mathfrak{x}), \tag{10}$$

where

$$\mathfrak{x} = \nu\mathfrak{x}_1. \tag{10'}$$

If $\mathfrak{y}_1$ is a further generic point independent of $\mathfrak{x}_1$, then

$$\vartheta_{\mathfrak{d}}(\mathfrak{x}, \mathfrak{y})\ (say) = \frac{\varphi_{\mathfrak{d}}(\mathfrak{x}_1)\,\varphi_{\mathfrak{o}}(\mathfrak{y}_1)}{\varphi_{\mathfrak{o}}(\mathfrak{x}_1 + \mathfrak{y}_1)} \in \mathfrak{K}(\mathfrak{x}, \mathfrak{y}) \tag{11}$$

where

$$\mathfrak{y} = \nu\mathfrak{y}_1 \tag{11'}$$

By (3) and (8) we have

$$\varphi_{\mathfrak{d}}(\mathfrak{x}_1 + \mathfrak{e}) = \chi(\mathfrak{d}, \mathfrak{e})\,\varphi_{\mathfrak{d}}(\mathfrak{x}_1) \tag{12}$$

for any solution $\mathfrak{e}$ of $\nu\mathfrak{e} = \mathfrak{o}$. Hence if (9) holds the expression (10) is unchanged when $\mathfrak{x}_1$ is replaced by $\mathfrak{x}_1 + \mathfrak{e}$. This proves the first part of the Lemma. The second part is proved similarly.

It is convenient to introduce the terms invariant system and invariance class. An invariant system is defined to be a set $\{A_{\mathfrak{d}}\}$ of non-zero elements of $\bar{\mathfrak{K}}$, one for each solution $\mathfrak{d}$ of $\nu\mathfrak{d} = \mathfrak{o}$, with $A_{\mathfrak{o}} = 1$ and with the property that

$$(A_{\mathfrak{d}_1})^{m_1} \cdots (A_{\mathfrak{d}_s})^{m_s} \in \mathfrak{K} \tag{13}$$

whenever $m_1, \ldots, m_s$ are such that

$$m_1\mathfrak{d}_1 + \cdots + m_s\mathfrak{d}_s = \mathfrak{o}. \tag{14}$$

Two invariant systems $\{A'_{\mathfrak{d}}\}$, $\{A''_{\mathfrak{d}}\}$ are defined to be in the same invariance class S if there are $B_{\mathfrak{d}} \in \mathfrak{K}$ such that

$$A''_{\mathfrak{d}} = B_{\mathfrak{d}} A'_{\mathfrak{d}}. \tag{15}$$

Multiplication of invariant systems and invariance classes is by the rule

$$\{A'_{\mathfrak{d}}\}\ \{A''_{\mathfrak{d}}\} = \{A'_{\mathfrak{d}} A''_{\mathfrak{d}}\}.$$

We may express the invariance condition (13), (14) in another way. If ϱ is any automorphism of $\bar{\mathfrak{K}}/\mathfrak{K}$ then

$$\varrho A_{\mathfrak{d}} = \psi_{\varrho}(\mathfrak{d})\, A_{\mathfrak{d}}, \tag{15'}$$

where $\psi_{\varrho}(\mathfrak{d})$ is a character on the group of the $\mathfrak{d}$ which depends on ϱ and on the invariance class of $\{A'_{\mathfrak{d}}\}$. The equivalence of the two formulations is readily verified.

As Hasse (4) remarks in a similar context, the invariance class to which the invariant system $\{A_{\mathfrak{d}}\}$ belongs is completely determined by the left hand side of (13) (for all $s, \mathfrak{d}_1, \ldots, \mathfrak{d}_s, m_1, \ldots, m_s$); that is the invariance classes can be specified by objects in $\mathfrak{K}$ (and not in $\bar{\mathfrak{K}}$, as we have done).

Condition (II) of Theorem I, namely that all the $\mathfrak{d}$ shall be defined over $\mathfrak{K}$, allows us to specify the sets of invariant systems and invariance classes completely. If $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ is a basis for the $\mathfrak{d}$ with orders $e_1, \ldots, e_s$ respectively, then we may specify an invariant system by elements[12]) $a_1, \ldots, a_s \in \mathfrak{K}^*$ on putting

$$(A_{\mathfrak{e}_j})^{e_j} = a_j \qquad (1 \leq j \leq s)$$

[12]) $\mathfrak{K}^*$ is the multiplicative group of non-zero elements of $\mathfrak{K}$.

and then

$$(15'') \qquad A_{\mathfrak{d}} = A_{\mathfrak{e}_1}^{u_1} \cdots A_{\mathfrak{e}_s}^{u_s}$$

for

$$(15''') \qquad \mathfrak{d} = u_1\mathfrak{e}_1 + \cdots + u_s\mathfrak{e}_s \qquad (0 \leqq u_j < e_j).$$

Since the $\mathfrak{d}$ are in $\mathfrak{K}$, this is an invariant system. Clearly every invariant system is in the same class as one of those of the special type above specified. Further, $(a_1', \ldots, a_s')$ and $(a_1'', \ldots, a_s'')$ give rise to invariant systems in the same class if and only if[13])

$$(a_j')^{-1} a_j'' \in (\mathfrak{K}^*)^{e_j} \qquad (1 \leqq j \leqq s).$$

Hence the group of invariance classes is isomorphic to the direct product of the groups

$$\mathfrak{K}^*/(\mathfrak{K}^*)^{e_j} \qquad (1 \leqq j \leqq s).$$

The following result is due in essence to Weil.

Lemma 3. *There is a homomorphism of the group $\mathfrak{G}$ of points of C defined over $\mathfrak{K}$ into the group of invariance classes defined as follows.*

Let $\mathfrak{a} \in \mathfrak{G}$ and let $\mathfrak{a}'$ be any point of C with $\nu\mathfrak{a}' = \mathfrak{a}$, so $\mathfrak{a}'$ is not in general defined over $\mathfrak{K}$. Then the invariance class corresponding to $\mathfrak{a}$ is that to which the invariant system $\{\varphi_{\mathfrak{b}}(\mathfrak{a}')\}$ belongs. This invariance class is independent both of the choice of $\mathfrak{a}'$ as a solution of $\nu\mathfrak{a}' = \mathfrak{a}$ and of the choice of $g(\mathfrak{x})$ in (8), provided that all the $\varphi_{\mathfrak{b}}(\mathfrak{a}')$ are defined and not 0 or ∞.

The kernel of the homomorphism is just $\nu\mathfrak{G}$.

First, $\{\varphi_{\mathfrak{b}}(\mathfrak{a}')\}$ is an invariant system. For the fact that (10), subject to (9) is a function of $\mathfrak{x} = \nu\mathfrak{x}_1$ defined over $\mathfrak{K}$ shows that, when $\mathfrak{x}_1$ is specialized to $\mathfrak{a}'$, so $\mathfrak{x}$ is specialized to $\mathfrak{a}$, in the left-hand side of (10) we get an element of $\mathfrak{K}$. This shows that the condition (13) with $A_{\mathfrak{b}} = \varphi_{\mathfrak{b}}(\mathfrak{a}')$ is satisfied.

Secondly, the invariance class obtained is independent of the choice of $g(\mathfrak{x})$ in (8). For the replacement of $g(\mathfrak{x})$ by another function $g'(\mathfrak{x})$ multiplies $\varphi_{\mathfrak{b}}(\mathfrak{a}')$ by the value $B_{\mathfrak{b}}$ (say) taken by

$$\frac{g'(\mathfrak{x})\, g(\mathfrak{x} - \mathfrak{b})}{g'(\mathfrak{x} - \mathfrak{b})\, g(\mathfrak{x})}$$

at $\mathfrak{x} = \mathfrak{a}$, so $B_{\mathfrak{b}} \in \mathfrak{K}$.

Thirdly the invariance class obtained is independent of the choice of $\mathfrak{a}'$. For let $\nu\mathfrak{a}'' = \mathfrak{a}$. Then $\mathfrak{a}'' = \mathfrak{a}' + \mathfrak{e}$, where $\nu\mathfrak{e} = \mathfrak{o}$. Hence, by (12), we have $\varphi_{\mathfrak{b}}(\mathfrak{a}'') = \chi(\mathfrak{b}, \mathfrak{e})\, \varphi_{\mathfrak{b}}(\mathfrak{a}')$, where $\chi(\mathfrak{b}, \mathfrak{e}) \in \mathfrak{K}$ by (7).

Fourthly, the mapping is a homomorphism. For suppose that $\mathfrak{a} + \mathfrak{b} = \mathfrak{c}$. Let $\nu\mathfrak{a}' = \mathfrak{a}$, $\nu\mathfrak{b}' = \mathfrak{b}$, so $\nu\mathfrak{c}' = \mathfrak{c}$ where $\mathfrak{c}' = \mathfrak{a}' + \mathfrak{b}'$. By the second part of Lemma 2 we have

$$\frac{\varphi_{\mathfrak{b}}(\mathfrak{a}')\, \varphi_{\mathfrak{b}}(\mathfrak{b}')}{\varphi_{\mathfrak{b}}(\mathfrak{c}')} = \vartheta_{\mathfrak{b}}(\mathfrak{a}, \mathfrak{b}) \in \mathfrak{K}.$$

Hence the invariance class of $\{\varphi_{\mathfrak{b}}(\mathfrak{c}')\}$ is the product of those of $\{\varphi_{\mathfrak{b}}(\mathfrak{a}')\}$ and $\{\varphi_{\mathfrak{b}}(\mathfrak{a}'')\}$.

Fifthly, if $\mathfrak{a} \in \nu\mathfrak{G}$ then the invariance class of $\{\varphi_{\mathfrak{b}}(\mathfrak{a}')\}$ is the unit class[14]). For we may suppose that $\mathfrak{a}'$ is defined over $\mathfrak{K}$, and then $\varphi_{\mathfrak{b}}(\mathfrak{a}') \in \mathfrak{K}$.

---

[13]) We recollect that condition (II) implies that the e_j-th roots of 1 are in $\mathfrak{K}$.

[14]) i. e. $\varphi_{\mathfrak{b}}(\mathfrak{a}') \in \mathfrak{K}$ for all $\mathfrak{b}$.

Lastly, if $\{\varphi_{\mathfrak{d}}(\mathfrak{a}')\}$, is the unit class then $\mathfrak{a}' \in \mathfrak{G}$, and so $\nu\mathfrak{a}' = \mathfrak{a} \in \nu\mathfrak{G}$. For, if not, there is an automorphism ϱ of $\bar{\mathfrak{K}}/\mathfrak{K}$ such that $\varrho\mathfrak{a}' \neq \mathfrak{a}'$. Then $\varrho\mathfrak{a}' = \mathfrak{a}' + \mathfrak{e}$ for some solution $\mathfrak{e} \neq \mathfrak{o}$ of $\nu\mathfrak{e} = \mathfrak{o}$, since $\nu(\varrho\mathfrak{a}') = \varrho(\nu\mathfrak{a}') = \varrho\mathfrak{a} = \mathfrak{a} = \nu\mathfrak{a}'$. Hence

$$\varrho\varphi_{\mathfrak{d}}(\mathfrak{a}') = \varphi_{\mathfrak{d}}(\varrho\mathfrak{a}) = \chi'(\mathfrak{d}, \mathfrak{e})\, \varphi_{\mathfrak{d}}(\mathfrak{a}').$$

Since the $\varphi_{\mathfrak{d}}(\mathfrak{a}')$ are assumed to be in $\mathfrak{K}$, we must have $\chi(\mathfrak{d}, \mathfrak{e}) = 1$ for all $\mathfrak{d}$: and this can only happen if $\mathfrak{e} = \mathfrak{o}$, a contradiction.

The notion of ν-covering was defined in § 1. We have

Lemma 4. *There is a 1 — 1 correspondence between the invariance classes and the ν-coverings.*

Suppose, first, that an invariance class S is given. Let $\{A_{\mathfrak{d}}\}$ be an invariant system in S. Then there is an obvious birational correspondence defined over $\bar{\mathfrak{K}}$ between the points $\mathfrak{x}_1$ on C and the points[15]) $\mathfrak{X} = (\mathfrak{x}, t_{\mathfrak{d}_1}, \ldots, t_{\mathfrak{d}_{N-1}})$, where

$$(16) \qquad \begin{cases} \mathfrak{x} = \nu\mathfrak{x}_1 \\ \varphi_{\mathfrak{d}}(\mathfrak{x}_1) = A_{\mathfrak{d}} t_{\mathfrak{d}} \end{cases}$$

and $\mathfrak{d}_1, \ldots, \mathfrak{d}_{N-1}$ is all the solutions of $\nu\mathfrak{d} = \mathfrak{o}$, $\mathfrak{d} \neq \mathfrak{o}$. Further, the locus $\mathcal{D}$ of $\mathfrak{X}$, when $\mathfrak{x}_1$ is a generic point of C, is defined over $\mathfrak{K}$. For let ϱ be any automorphism of $\bar{\mathfrak{K}}/\mathfrak{K}$, and let

$$\varrho A_{\mathfrak{d}} = \psi_{\mathfrak{d}}(\mathfrak{d})\, A_{\mathfrak{d}}.$$

Here, as remarked above (cf. (15′)), $\psi_{\varrho}(\mathfrak{d})$ is a character on the group of the $\mathfrak{d}$, and so by the non-degeneracy of the bilinear form $\chi(\mathfrak{d}, \mathfrak{e})$ is of the shape

$$\psi_{\varrho}(\mathfrak{d}) = \chi(\mathfrak{d}, \mathfrak{e}_{\varrho})$$

for some solution $\mathfrak{e}_{\varrho}$ of $\nu\mathfrak{e}_{\varrho} = \mathfrak{o}$. Hence (16) is equivalent to

$$(16') \qquad \begin{cases} \mathfrak{x} = \mathfrak{x}_1' \\ \varphi_{\mathfrak{d}}(\mathfrak{x}_1') = (\varrho A_{\mathfrak{d}})\, t_{\mathfrak{d}}, \end{cases}$$

where $\mathfrak{x}_1' = \mathfrak{x}_1 + \mathfrak{e}_{\varrho}$ is a generic point of C when $\mathfrak{x}_1$ is, that is[16]) the locus of $\mathfrak{X}$ is unchanged by the action of ϱ.

Clearly the choice of another invariant system $\{A_{\mathfrak{d}}'\}$ in the same class as $\{A_{\mathfrak{d}}\}$ gives the same covering in the sense defined in § 1. For $A_{\mathfrak{d}}' = B_{\mathfrak{d}} A_{\mathfrak{d}}$, where $B_{\mathfrak{d}} \in \mathfrak{K}$, so there is a birational correspondence defined over $\mathfrak{K}$ between the corresponding curves $\mathcal{D}$, $\mathcal{D}'$ given, in an obvious notation, by $B_{\mathfrak{d}} t_{\mathfrak{d}}' = t_{\mathfrak{d}}$. Suppose now that a covering

$$(17) \qquad U: \quad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{D} \arrow[ur] & \end{tikzcd}$$

is given, with corresponding generic points

$$(17') \qquad \begin{tikzcd} \mathfrak{x}_1 \arrow[r] \arrow[d, leftrightarrow] & \nu\mathfrak{x}_1 = \mathfrak{x}. \\ \mathfrak{X} \arrow[ur] & \end{tikzcd}$$

---

[15]) The notation means that the co-ordinates of $\mathfrak{X}$ are those of $\mathfrak{x}$ together with $t_{\mathfrak{d}_1}, \ldots, t_{\mathfrak{d}_{N-1}}$.

[16]) We recollect that $\mathfrak{K}$ has characteristic 0, so inseparability cannot occur.

Here the correspondence $\mathfrak{x}_1 \longleftrightarrow \mathfrak{X}$ is defined over $\bar{\mathfrak{K}}$ but the other maps are defined over $\mathfrak{K}$. The inverse image of a divisor $(\mathfrak{o})^{-1}(\mathfrak{d})$ under the map $\mathfrak{X} \to \mathfrak{x}$ is a divisor on $\mathcal{D} = \mathcal{D}(U)$ which is defined over $\mathfrak{K}$ since $\mathfrak{d}$ is and which is linearly equivalent to 0. Hence there is a function $F_{\mathfrak{d}}(\mathfrak{X})$ defined over $\mathfrak{K}$ with this divisor. It follows that[17])

$$\frac{\Phi_{\mathfrak{d}}(\mathfrak{x}_1(\mathfrak{X}))}{F_{\mathfrak{d}}(\mathfrak{X})} = A_{\mathfrak{d}} \tag{18}$$

is independent of $\mathfrak{X}$, i. e. is in $\bar{\mathfrak{K}}$. Let ϱ be any automorphism of $\bar{\mathfrak{K}}/\mathfrak{K}$. We may extend ϱ to $\bar{\mathfrak{K}}(\mathfrak{X})$ by the convention $\varrho\mathfrak{X} = \mathfrak{X}$ and then $\varrho\mathfrak{x}_1(\mathfrak{X}) = \mathfrak{x}_1(\mathfrak{X}) + \mathfrak{e}_\varrho$, for some $\mathfrak{e}_\varrho$ on C with $\nu\mathfrak{e}_\varrho = \mathfrak{o}$, by the commutativity of (17). Hence

$$\varrho A_{\mathfrak{d}} = \chi(\mathfrak{d}, \mathfrak{e}_\varrho)\, A_{\mathfrak{d}}.$$

This implies that the $A_{\mathfrak{d}}$ are an invariant system, i. e. satisfy (15). On putting $t_{\mathfrak{d}} = F_{\mathfrak{d}}(\mathfrak{X}(\mathfrak{x}_1))$ we have a set of equations for $\mathcal{D}$ of the shape (16).

Finally, we note that the isomorphism $\mathfrak{x}_1 \to \mathfrak{x}_1 + \mathfrak{d}$ of C gives an isomorphism $\mathfrak{X} \to \mathfrak{X} + \mathfrak{d}$ of $\mathcal{D}$ defined over $\mathfrak{K}$. Thus we have a $1-1$ correspondence between invariance classes and coverings in the sense of § 1 (cf. (4) of § 1).

Corollary. *A necessary and sufficient condition that the ν-covering is one which corresponds to an element of $\mathfrak{G}/\nu\mathfrak{G}$ in the sense of Lemma 3 is that there should be a point on $\mathcal{D}$ defined over $\mathfrak{K}$.*

This follows at once from (16) and the enunciation of Lemma 3.

Lemma 5. *Let $\mathfrak{x}_1', \mathfrak{x}', \mathfrak{X}'$ and $\mathfrak{x}_1'', \mathfrak{x}'', \mathfrak{X}''$ be two trios of corresponding points for the same covering (17). Then the map $(\mathfrak{x}_1', \mathfrak{x}_1'') \to \mathfrak{z} = \mathfrak{x}_1' - \mathfrak{x}_1''$, where $\mathfrak{z}$ is on C, gives rise to a map $(\mathfrak{X}', \mathfrak{X}'') \to \mathfrak{z}$, defined over $\mathfrak{K}$, of pairs of points on $\mathcal{D}$ onto C. If $\mathfrak{A}'$ is a point of $\mathcal{D}$ defined over $\mathfrak{K}$, the map*

$$(\mathfrak{A}', \mathfrak{X}'') \to \mathfrak{z} \tag{19}$$

is a birational correspondence defined over $\mathfrak{K}$ between generic points $\mathfrak{X}''$, $\mathfrak{z}$ of $\mathcal{D}$, C.

Note. If $(\mathfrak{X}', \mathfrak{X}'') \to \mathfrak{z}$ we shall often write

$$\mathfrak{X}' - \mathfrak{X}'' = \mathfrak{z}$$

or

$$\mathfrak{X}' = \mathfrak{X}'' + \mathfrak{z}.$$

In general the map just defined depends not merely on the curve $\mathcal{D}$ but also on the particular covering U.

For, in an obvious notation (cf. (16)), we have

$$\varphi_{\mathfrak{d}}(\mathfrak{x}_1') = A_{\mathfrak{d}} t_{\mathfrak{d}}',$$
$$\varphi_{\mathfrak{d}}(\mathfrak{x}_1'') = A_{\mathfrak{d}} t_{\mathfrak{d}}''.$$

Hence, by (11),

$$\varphi_{\mathfrak{d}}(\mathfrak{z}) = \varphi_{\mathfrak{d}}(\mathfrak{x}_1' - \mathfrak{x}_1'') = \frac{t_{\mathfrak{d}}'}{t_{\mathfrak{d}}''\, \vartheta_{\mathfrak{d}}(\mathfrak{x}' - \mathfrak{x}'', \mathfrak{x}'')}.$$

These equations, together with

$$\nu\mathfrak{z} = \mathfrak{x}' - \mathfrak{x}''$$

determine $\mathfrak{z}$ completely in terms of $\mathfrak{X}'$, $\mathfrak{X}''$; and they are defined over $\mathfrak{K}$. The rest of the lemma is now trivial.

---

[17]) $\mathfrak{x}_1(\mathfrak{X})$ is the point $\mathfrak{x}_1$ which corresponds to $\mathfrak{X}$ in (17′). We shall use similar notation in future without special comment.

The following lemma shows how the structure of the group of invariance classes is reflected in the ν-coverings.

Lemma 6. *Let S', S'' and $S = S'S''$ be invariance systems, let U', U'', U be the corresponding coverings and put $\mathcal{D}' = \mathcal{D}(U')$, $\mathcal{D}'' = \mathcal{D}(U'')$, $\mathcal{D} = \mathcal{D}(U)$. Let $\mathfrak{x}_1', \mathfrak{X}', \mathfrak{x}'$; $\mathfrak{x}_1'', \mathfrak{X}'', \mathfrak{x}''$; $\mathfrak{x}_1, \mathfrak{X}, \mathfrak{x}$ be corresponding trios of generic points on the figures corresponding to* (17). *Then the map*

$$(\mathfrak{x}_1', \mathfrak{x}_1'') \to \mathfrak{x}_1' + \mathfrak{x}_1'' = \mathfrak{x}_1$$

gives rise to a map

$$(\mathfrak{X}', \mathfrak{X}'') \to \mathfrak{X}$$

which is defined over $\mathfrak{K}$.

Note. If $\mathfrak{A}', \mathfrak{A}''$ are points on $\mathcal{D}', \mathcal{D}''$ respectively, we denote by $\mathfrak{A}' + \mathfrak{A}''$ the image of $(\mathfrak{A}', \mathfrak{A}'')$ in the above map. This notation is slightly ambiguous because the map of Lemma 6 depends not merely on the coverings U', U'', U but also on the particular choice of maps $\mathfrak{X}' \longleftrightarrow \mathfrak{x}_1'$ etc. If $\mathfrak{A}^0$ is another determination of $\mathfrak{A}' + \mathfrak{A}''$, then $\nu(\mathfrak{A} - \mathfrak{A}^0) = \mathfrak{o}$.

We use the representation (16). In an obvious notation, we may suppose without loss of generality that

$$A_\mathfrak{b} = A_\mathfrak{b}' A_\mathfrak{b}''.$$

By (11) we have

$$t_\mathfrak{b} = \vartheta_\mathfrak{b}(\mathfrak{x}', \mathfrak{x}'')\, t_\mathfrak{b}' t_\mathfrak{b}''.$$

This is a set of equations defined over $\mathfrak{K}$ for the required transformation.

§ 3. Extension of coverings.

Throughout this section, $\mathfrak{K}$ is a field of characteristic 0 and conditions (I), (II) of Theorem I are assumed to hold. We shall be investigating the situation

$$(1) \qquad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{S} \arrow[r] & \mathcal{D} \arrow[ur] & \end{tikzcd}$$

with corresponding generic points

$$(1') \qquad \begin{tikzcd} \mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x}_1 = \nu\mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x} = \nu\mathfrak{x}_1. \\ \mathfrak{X} \arrow[r] & \mathfrak{X} \arrow[ur] & \end{tikzcd}$$

Here the birational correspondences are only defined over $\bar{\mathfrak{K}}$ but the other maps are defined over $\mathfrak{K}$.

We shall sometimes use notation based on (1), (1′) without special reference.

Lemma 7. *Suppose that* (1) *exists. Then for each solution $\mathfrak{d}$ of $\nu\mathfrak{d} = \mathfrak{o}$ there exists on $\mathcal{D}$ a divisor $\mathfrak{D}(\mathfrak{d})$ defined over $\mathfrak{K}$, which is linearly equivalent to $(\mathfrak{X})^{-1}(\mathfrak{X} + \mathfrak{d})$.*

Let $\mathfrak{O}$ be any point on $\mathcal{D}$ defined over $\bar{\mathfrak{K}}$, and let $\mathfrak{L}$ be a finite normal extension of $\mathfrak{K}$ over which $\mathfrak{O}$ is defined. Then the inverse image[18]) of $(\mathfrak{O})^{-1}(\mathfrak{O}+\mathfrak{d})$ on $\mathcal{S}$ under the map $\mathbf{X}\to\mathfrak{X}$ is linearly equivalent to zero; and so there exists a function[19]) $\psi(\mathbf{X})\in\mathfrak{K}(\mathbf{X})$ with this inverse image as divisor. If ϱ is any automorphism of $\mathfrak{L}/\mathfrak{K}$ we may extend it to $\mathfrak{L}(\mathbf{X})$ by putting $\varrho\mathbf{X}=\mathbf{X}$. The divisor of $\varrho\psi(\mathbf{X})$ is the inverse image of $\varrho\{(\mathfrak{O})^{-1}(\mathfrak{O}+\mathfrak{d})\}=(\varrho\mathfrak{O})^{-1}(\varrho\mathfrak{O}+\mathfrak{d})$, since $\varrho\mathfrak{d}=\mathfrak{d}$ by condition (II) of Theorem I. Since $(\mathfrak{O})^{-1}(\mathfrak{O}+\mathfrak{d})$ and $(\varrho\mathfrak{O})^{-1}(\varrho\mathfrak{O}+\mathfrak{d})$ are linearly equivalent, then we have

$$\frac{\varrho\psi(\mathbf{X})}{\psi(\mathbf{X})}=\Psi_\varrho(\mathfrak{X})\in\mathfrak{L}(\mathfrak{X}). \tag{2}$$

By (2) we have

$$\Psi_\varrho(\mathfrak{X})\,\varrho\Psi_{\varrho^{-1}\sigma}(\mathfrak{X})=\Psi_\varrho(\mathfrak{X})$$

for any two automorphisms ϱ,σ of $\mathfrak{L}(\mathbf{X})/\mathfrak{K}(\mathbf{X})$. Hence by Noether's extension of Hilbert's Satz 90 there is a $\Theta(\mathfrak{X})\in\mathfrak{L}(\mathfrak{X})$ such that

$$\Psi_\varrho(\mathfrak{X})=\frac{\varrho\Theta(\mathfrak{X})}{\Theta(\mathfrak{X})}. \tag{3}$$

Let the divisor of $\Theta(\mathfrak{X})$ be $\mathfrak{O}^{-1}(\mathfrak{O}+\mathfrak{d})\,\mathfrak{D}(\mathfrak{d})$. Since the divisor of $\Psi_\varrho(\mathfrak{X})$ is $(\varrho\mathfrak{O})^{-1}(\varrho\mathfrak{O}+\mathfrak{d})(\mathfrak{O})(\mathfrak{O}+\mathfrak{d})^{-1}$ by construction, it follows from (3) that

$$\varrho\mathfrak{D}(\mathfrak{d})=\mathfrak{D}(\mathfrak{d}).$$

Hence $\mathfrak{D}(\mathfrak{d})$ is defined over $\mathfrak{K}$. This completes the proof of the Lemma.

The following propositions merely assume the existence of $\mathcal{D}$ and $\mathfrak{D}(\mathfrak{d})$ and not necessarily the existence of $\mathcal{S}$.

Proposition 1. *Let $\mathfrak{e}_1,\ldots,\mathfrak{e}_s$ be a basis for the solutions of $\nu\mathfrak{x}=\mathfrak{o}$ with respective orders $e_1,\ldots,e_s$. Then if*

$$\mathfrak{d}=\sum_{j=1}^{s}u_j\mathfrak{e}_j\quad(0\leqq u_j<e_j), \tag{4}$$

we may put

$$\mathfrak{D}(\mathfrak{d})=\prod_j(\mathfrak{D}(\mathfrak{e}_j))^{u_j}. \tag{5}$$

This is clear, since the $\mathfrak{e}_j$ are all defined over $\mathfrak{K}$.

Proposition 2. *Let d be the order of $\mathfrak{d}$ in the group of solutions of $\nu\mathfrak{x}=\mathfrak{o}$. Then there is an $\Omega_{\mathfrak{d}}(\mathfrak{X})\in\mathfrak{K}(\mathfrak{X})$ having divisor $(\mathfrak{D}(\mathfrak{d}))^d$.*

For $(\mathfrak{O})^{-d}(\mathfrak{O}+\mathfrak{d})^d$ is linearly equivalent to 0.

Proposition 3. *Consider the maps*

$$\begin{tikzcd} C \arrow[r] & C \arrow[r] \arrow[d, dashed, leftrightarrow] & C \\ & \mathcal{D} \arrow[ur] & \end{tikzcd} \tag{6}$$

[18]) For notation, see note after the enunciation of Lemma 5.

[19]) $\mathfrak{K}(\mathbf{X})$ is the field of functions of $\mathbf{X}$ defined over $\mathfrak{K}$.

with corresponding generic points

$$\begin{tikzcd} \mathfrak{x}_2 \arrow[r] & \mathfrak{x}_1 = \nu\mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x} = \nu\mathfrak{x}_1 . \\ & \mathfrak{X} \arrow[ur] & \end{tikzcd} \tag{7}$$

Then, in the notation of the previous proposition

$$\Omega_{\mathfrak{d}}(\mathfrak{X}(\mathfrak{x}_2)) = \{\omega_{\mathfrak{d}}(\mathfrak{x}_2)\}^d, \tag{8}$$

where $\omega_{\mathfrak{d}}(\mathfrak{x}_2) \in \bar{\mathfrak{K}}(\mathfrak{x}_2)$. *Further,*

$$\omega_{\mathfrak{d}}(\mathfrak{x}_2 + \mathfrak{e}) = \chi(\mathfrak{d}, \mathfrak{e})\, \omega_{\mathfrak{d}}(\mathfrak{x}_2). \tag{9}$$

For the inverse image on $\mathcal{C}$ of the divisor $\mathfrak{D}(\mathfrak{d})$ on $\mathcal{D}$ under $\mathfrak{x}_2 \to \mathfrak{X}$ is linearly equivalent to the inverse image of $(\mathfrak{X})^{-1}(\mathfrak{X} + \mathfrak{d})$ and so to zero. Hence there exists a function $\omega_{\mathfrak{d}}(\mathfrak{x}_2) \in \bar{\mathfrak{K}}(\mathfrak{x}_2)$ with this divisor; and we may suppose that (8) holds, on incorporating a constant factor in $\omega_{\mathfrak{d}}(\mathfrak{x}_2)$. If $\Phi_{\mathfrak{d}}(\mathfrak{x})$ is the function defined in § 2 and if $\mathfrak{O}$ is the image of $\mathfrak{o}$ under the map $\mathfrak{x}_1 \to \mathfrak{X}$, comparison of zeros and poles shows that

$$\frac{\omega_{\mathfrak{d}}(\mathfrak{x}_2)\, \zeta_{\mathfrak{d}}(\mathfrak{X}(\mathfrak{x}_2))}{\Phi_{\mathfrak{d}}(\mathfrak{x}_2)} \in \bar{\mathfrak{K}},$$

where $\zeta_{\mathfrak{d}}(\mathfrak{X}) \in \bar{\mathfrak{K}}(\mathfrak{X})$ is any function on $\mathcal{D}$ with divisor $(\mathfrak{O})^{-1}(\mathfrak{O} + \mathfrak{d})\, \{\mathfrak{D}(\mathfrak{d})\}^{-1}$.

Then (9) follows from (3) of § 2 and the fact that $\mathfrak{X}(\mathfrak{x}_2 + \mathfrak{e}) = \mathfrak{X}(\mathfrak{x}_2)$, which follows from (7).

Lemma 8. *Suppose that U is a ν-covering of $\mathcal{C}$ and that for every $\mathfrak{d}$ with $\nu\mathfrak{d} = \mathfrak{o}$ there is a divisor $\mathfrak{D}(\mathfrak{d})$ on $\mathcal{D}(U)$ which is defined over $\mathfrak{K}$ and which is linearly equivalent to $(\mathfrak{X})^{-1}(\mathfrak{X} + \mathfrak{d})$. Then U is extensible to a ν^2-covering of $\mathcal{C}$.*

There is then a $1-1$ correspondence between the invariance classes and the ν^2-coverings $\mathcal{V}$ which extend U. This correspondence is set up as follows:

Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ be a basis for the solutions of $\nu\mathfrak{x} = \mathfrak{o}$ with orders $e_1, \ldots, e_s$. If

$$\mathfrak{d} = \sum_{j=1}^{s} u_j \mathfrak{e}_j \qquad (0 \leqq u_j < e_j) \tag{10}$$

put

$$\alpha_{\mathfrak{d}}(\mathfrak{x}_2) = \prod_j \{\omega_{\mathfrak{e}_j}(\mathfrak{x}_2)\}^{u_j}, \tag{11}$$

where $\omega_{\mathfrak{e}_j}(\mathfrak{x}_2)$ is given by (8). Let the points $\mathfrak{X}$ on $\mathcal{D} = \mathcal{D}(U)$ and the points $T_{\mathfrak{d}}$ ($\nu\mathfrak{d} = \mathfrak{o}$, $\mathfrak{d} \neq \mathfrak{o}$) on the projective line be given by

$$\begin{cases} \mathfrak{X} = \mathfrak{X}(\mathfrak{x}_2) \\ A_{\mathfrak{d}}\, T_{\mathfrak{d}} = \alpha_{\mathfrak{d}}(\mathfrak{x}_2), \end{cases} \tag{12}$$

where $\{A_{\mathfrak{d}}\}$ is an invariant system in a given invariance class. Then the locus of the point $\mathbf{X} = (\mathfrak{X}, T_{\mathfrak{d}_1}, \ldots, T_{\mathfrak{d}_{N-1}})$ is a curve $\mathcal{E}$ defined over $\mathfrak{K}$ and

$$\mathcal{V}: \quad \begin{tikzcd} \mathfrak{x}_2 \arrow[r] \arrow[d, leftrightarrow] & \nu\mathfrak{x}_2 = \mathfrak{x}_1 \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{x} = \nu\mathfrak{x}_1 \\ \mathbf{X} \arrow[r] & \mathfrak{X} \arrow[ur] & \end{tikzcd}$$

is an extension of the original covering, where $\mathfrak{x}_2 \longleftrightarrow \mathbf{X}$ is given by (12).

Further, all extensions of the original covering are given in this way.

In the first place, by (9), (10), (11) we have

$$\alpha_{\mathfrak{b}}(\mathfrak{x}_2 + \mathfrak{e}) = \chi(\mathfrak{b}, \mathfrak{e})\, \alpha_{\mathfrak{b}}(\mathfrak{x}_2) \tag{12'}$$

whenever $\nu\mathfrak{e} = \mathfrak{o}$.

We next show that the equations (12) give a curve $\mathfrak{S}$ defined over $\mathfrak{K}$. Let ϱ be any automorphism of $\bar{\mathfrak{K}}/\mathfrak{K}$. We extend ϱ to $\bar{\mathfrak{K}}(\mathfrak{X})$ by putting $\varrho\mathfrak{X} = \mathfrak{X}$ and then further extend ϱ in some fixed way to the finite algebraic extension $\bar{\mathfrak{K}}(\mathfrak{x}_2)$ of $\bar{\mathfrak{K}}(\mathfrak{X})$. By (8) we have

$$\varrho\omega_{\mathfrak{b}}(\mathfrak{x}_2) = r_{\mathfrak{b}}\, \omega_{\mathfrak{b}}(\mathfrak{x}_2)$$

where $r_{\mathfrak{b}}$ is a d-th root of 1 (d being the order of $\mathfrak{b}$). In particular, if (10) holds we have[20], by (11),

$$\begin{aligned}\varrho\alpha_{\mathfrak{b}}(\mathfrak{x}_2) &= \Big\{\prod_j r_{e_j}^{u_j}\Big\}\, \alpha_{\mathfrak{b}}(\mathfrak{x}_2)\\ &= \chi'(\mathfrak{b})\, \alpha_{\mathfrak{b}}(\mathfrak{x}_2) \qquad \text{(say)},\end{aligned}$$

where $\chi'(\mathfrak{b})$ is a character on the $\mathfrak{b}$, since $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ is a basis for the $\mathfrak{b}$. But any character $\chi'(\mathfrak{b})$ can be put in the shape

$$\chi'(\mathfrak{b}) = \chi(\mathfrak{b}, \mathfrak{e}'_\varrho)$$

for some $\mathfrak{e}'_\varrho$ depending only on χ', i. e. on ϱ. Further, as in the proof of Lemma 4, if $\{A_{\mathfrak{b}}\}$ is an invariant system we have

$$\varrho A_{\mathfrak{b}} = \chi(\mathfrak{b}, \mathfrak{e}_\varrho)$$

for some $\mathfrak{e}_\varrho$. We conclude by (12′) that (12) is equivalent to

$$\begin{cases} \mathfrak{X} = \mathfrak{X}(\mathfrak{x}'_2) \\ (\varrho A_{\mathfrak{b}})\, T_{\mathfrak{b}} = \alpha_{\mathfrak{b}}(\mathfrak{x}'_2), \end{cases}$$

where $\mathfrak{x}'_2 = \mathfrak{x}_2 + \mathfrak{e}_\varrho - \mathfrak{e}'_\varrho$. Hence the locus of $\mathbf{X} = (\mathfrak{X}, T_{\mathfrak{b}_1}, \ldots, T_{\mathfrak{b}_{N-1}})$ is invariant under ϱ, and so is defined over $\mathfrak{K}$.

The rest of the proof of Lemma 8 is practically identical with that of Lemma 4 and will be omitted. This concludes the proof of Lemma 8.

We note that the $1-1$ correspondence given by Lemma 8 is not a natural one, since the functions $\alpha_{\mathfrak{b}}(\mathfrak{x}_2)$ are not uniquely determined by U and $\mathfrak{b}$. We investigate this ambiguity and consider some further properties of the $\alpha_{\mathfrak{b}}(\mathfrak{x}_2)$.

Lemma 9. *Let $\alpha'_{\mathfrak{b}}(\mathfrak{x}_2)$ be functions on C defined over $\mathfrak{K}$ with the properties:*

(i) *the divisor of $\alpha'_{\mathfrak{b}}(\mathfrak{x}_2)$ is the inverse image under $\mathfrak{x}_2 \to \mathfrak{X}$ of a divisor $\mathfrak{D}'(\mathfrak{b})$ on $\mathfrak{D}$ which is defined over $\mathfrak{K}$ and linearly equivalent to $(\mathfrak{X})^{-1}(\mathfrak{X} + \mathfrak{b})$.*

$$\text{(ii)} \qquad \{\alpha'_{\mathfrak{b}_1}(\mathfrak{x}_2)\}^{m_1} \cdots \{\alpha'_{\mathfrak{b}_t}(\mathfrak{x}_2)\}^{m_t} \in \mathfrak{K}(\mathfrak{X}) \tag{13}$$

where $\mathfrak{X} = \mathfrak{X}(\mathfrak{x}_2)$, provided that $m_1\mathfrak{b}_1 + \cdots + m_t\mathfrak{b}_t = \mathfrak{o}$.

Then

$$\alpha'_{\mathfrak{b}}(\mathfrak{x}_2) = Y_{\mathfrak{b}}(\mathfrak{X}(\mathfrak{x}_2))\, A_{\mathfrak{b}}\, \alpha_{\mathfrak{b}}(\mathfrak{x}_2), \tag{14}$$

where $Y_{\mathfrak{b}}(\mathfrak{X}) \in \mathfrak{K}(\mathfrak{X})$ and $\{A_{\mathfrak{b}}\}$ is an invariant system. Conversely, if the $\alpha'_{\mathfrak{b}}(\mathfrak{x}_2)$ are of the form (14), *then* (i) *and* (ii) *are true.*

---

[20]) This is where we essentially use the condition (II) of Theorem I in the proof of Theorem II.

Since $(\mathfrak{X})^{-1}(\mathfrak{X}+\mathfrak{d})$ is linearly equivalent to $\mathfrak{D}(\mathfrak{d})$, there exist by (i) functions $Y_{\mathfrak{d}}(\mathfrak{X}) \in \mathfrak{K}(\mathfrak{X})$ with respective divisors $\mathfrak{D}'(\mathfrak{d})\{\mathfrak{D}(\mathfrak{d})\}^{-1}$. Hence

$$\frac{\alpha'_{\mathfrak{d}}(\mathfrak{x}_2)}{Y_{\mathfrak{d}}(\mathfrak{X}(\mathfrak{x}_2))\,\alpha_{\mathfrak{d}}(\mathfrak{x}_2)} = A_{\mathfrak{d}} \in \bar{\mathfrak{K}}$$

is independent of $\mathfrak{x}_2$. It follows from (ii) of the enunciation of the Lemma and the corresponding statement for the $\alpha_{\mathfrak{d}}(\mathfrak{x}_2)$ that $\{A_{\mathfrak{d}}\}$ is an invariant system. This proves the direct statement of the Lemma, and the converse is trivial.

In future we shall denote by $\alpha_{\mathfrak{d}}(\mathfrak{x}_2)$ any set of functions having the properties (i), (ii) of $\alpha'_{\mathfrak{d}}(\mathfrak{x}_2)$ in Lemma 9. By (14) the equation (12′) clearly continues to hold with this new definition of $\alpha_{\mathfrak{d}}(\mathfrak{x}_2)$.

Corollary 1. *Suppose that the sets of equations*

$$(15) \qquad \begin{cases} \mathfrak{X} = \mathfrak{X}(\mathfrak{x}_2) \\ T_{\mathfrak{d}} = \alpha_{\mathfrak{d}}(\mathfrak{x}_2) \qquad (\nu\mathfrak{d} = \mathfrak{o},\ \mathfrak{d} \neq \mathfrak{o}) \end{cases}$$

and

$$(16) \qquad \begin{cases} \mathfrak{X} = \mathfrak{X}(\mathfrak{x}_2) \\ T'_{\mathfrak{d}} = \alpha'_{\mathfrak{d}}(\mathfrak{x}_2) \qquad (\nu\mathfrak{d} = \mathfrak{o},\ \mathfrak{d} \neq \mathfrak{o}) \end{cases}$$

define the same extension by a ν^2-covering $\mathcal{V}$ of the given ν-covering U. Then the invariant system $\{A_{\mathfrak{d}}\}$ in (14) *can be taken to be* $\{1\}$. *And conversely if $\{A_{\mathfrak{d}}\}$ is in the same class as* $\{1\}$, *then* (15) *and* (16) *define the same ν^2-covering.*

For let (15) and (16) give curves $\mathcal{S}$, $\mathcal{S}'$ with corresponding generic points $\mathbf{X} = (\mathfrak{X}(\mathfrak{x}_2),\ T_{\mathfrak{d}}\ (\nu\mathfrak{d} = \mathfrak{o},\ \mathfrak{d} \neq \mathfrak{o}))$ and $\mathbf{X}' = (\mathfrak{X}(\mathfrak{x}_2),\ T'_{\mathfrak{d}}\ (\nu\mathfrak{d} = \mathfrak{o},\ \mathfrak{d} \neq \mathfrak{o}))$. By definition, $\mathcal{S}$ and $\mathcal{S}'$ with (15), (16) give the same extension[21]) if and only if the birational correspondence $\mathbf{X} \longleftrightarrow \mathbf{X}'$ mediated by $\mathfrak{x}_2$ is defined over $\mathfrak{K}$. If this is so, then $Y_{\mathfrak{d}}(\mathfrak{X}) = \dfrac{T_{\mathfrak{d}}(\mathfrak{x}_2)}{T'_{\mathfrak{d}}(\mathfrak{x}_2)}$ must be a function of $\mathfrak{X}$ on $\mathcal{D}$ which is defined over $\mathfrak{K}$. This proves the direct statement of the Lemma, and the converse is trivial.

Corollary 2. *To the pair consisting of a ν^2-covering $\mathcal{V}$ which extends a given ν-covering U and a point $\mathfrak{A}$ on $\mathcal{D} = \mathcal{D}(U)$ defined over $\mathfrak{K}$ there corresponds naturally an invariance class $\Sigma(\mathcal{V}, \mathfrak{A})$ defined as follows:*

Let $\mathfrak{a}_2$ be a point of the inverse image of $\mathfrak{A}$ under $\mathfrak{x}_2 \to \mathfrak{X}$. Suppose, as we may without loss of generality, that the curve $\mathcal{S} = \mathcal{S}(\mathcal{V})$ is given by (15). *Let $Y_{\mathfrak{d}}(\mathfrak{X})$ be any function of $\mathfrak{X}$ on $\mathcal{D}$, defined over $\mathfrak{K}$, such that*

$$\alpha'_{\mathfrak{d}}(\mathfrak{x}_2) = Y_{\mathfrak{d}}(\mathfrak{X}(\mathfrak{x}_2))\,\alpha_{\mathfrak{d}}(\mathfrak{x}_2)$$

---

[21]) We use here condition (II) of Theorem I. In the first place, $\mathcal{S}$, $\mathcal{S}'$ together with the given correspondences $\mathbf{X} \longleftrightarrow \mathfrak{x}_2$, $\mathbf{X}' \longleftrightarrow \mathfrak{x}_2$ define the same ν^2-covering if and only if there is a $\mathfrak{f}$ on C with $\nu^2\mathfrak{f} = \mathfrak{o}$ such that the birational correspondence between $\mathbf{X}$ and $\mathbf{X}'$ given by the commutativity of the following diagram

$$\begin{tikzcd} \mathbf{X}(\mathfrak{x}_2) \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \mathfrak{x}_2 \arrow[d, leftrightarrow] \\ \mathbf{X}'(\mathfrak{x}'_2) \arrow[r, leftrightarrow] & \mathfrak{x}'_2 = \mathfrak{x}_2 + \mathfrak{f} \end{tikzcd}$$

is defined over $\mathfrak{K}$. Because $\mathcal{S}$ and $\mathcal{S}'$ are both to give extensions of the same covering U, a glance at (1) shows that, in fact, $\nu\mathfrak{f} = \mathfrak{o}$. But now, by Condition (II) of Theorem I, the point $\mathfrak{f}$ is defined over $\mathfrak{K}$ and so also is the correspondence $\mathbf{X}'(\mathfrak{x}_2) \longleftrightarrow \mathbf{X}'(\mathfrak{x}_2 + \mathfrak{f})$ by the analogue of Lemma 5 with ν^2 for ν. This implies the statement of the text.

has neither zero nor pole at $\mathfrak{x}_2 = \mathfrak{a}_2$. *Then the required invariance class is that to which* $\{\alpha'_{\mathfrak{b}}(\mathfrak{a}_2)\}$ *belongs.*

A necessary and sufficient condition that $\Sigma(\mathcal{V}, \mathfrak{A}) = 1$ *is that there exists a point* $\mathbf{A}$ *on* $\mathcal{S}$, *defined over* $\mathfrak{K}$ *which is mapped onto* $\mathfrak{A}$ *by the map* $\mathcal{S} \to \mathcal{D}$.

Clearly the class just defined is independent of the choice of auxiliary function $Y_{\mathfrak{b}}(\mathfrak{X})$. Secondly, if $\mathfrak{a}'_2$ is another choice of $\mathfrak{a}_2$ we have $\mathfrak{a}'_2 = \mathfrak{a}_2 + \mathfrak{e}$, where $\nu\mathfrak{e} = \mathfrak{o}$, and so

$$\alpha_{\mathfrak{b}}(\mathfrak{a}'_2) = \chi(\mathfrak{d}, \mathfrak{e})\, \alpha_{\mathfrak{b}}(\mathfrak{a}_2) \tag{17}$$

by (12′). From (7) of § 2 it follows that $\{\alpha'_{\mathfrak{b}}(\mathfrak{a}'_2)\}$ and $\{\alpha'_{\mathfrak{b}}(\mathfrak{a}_2)\}$ are in the same class. Finally, by Corollary 1, the class of $\{\alpha'_{\mathfrak{b}}(\mathfrak{a}_2)\}$ is independent of the particular representation (1) of the extension.

It remains to prove the last sentence. Since the generic point $\mathbf{X} = \mathbf{X}(\mathfrak{x}_2)$ of $\mathcal{S}$ is given by (15), we have

$$\alpha_{\mathfrak{b}}(\mathfrak{x}_2(\mathbf{X})) \in \mathfrak{K}(\mathbf{X}). \tag{18}$$

If $\mathbf{A}$ is a point of $\mathcal{S}$ defined over $\mathfrak{K}$, it follows that $\alpha'_{\mathfrak{b}}(\mathfrak{x}_2(\mathbf{A})) \in \mathfrak{K}$, and so $\Sigma(\mathcal{V}, \mathfrak{A})$ is trivial. Conversely if $\mathbf{A}'$ is a point of $\mathcal{S}$ defined over $\bar{\mathfrak{K}}$ which maps onto $\mathfrak{A}$ and if $\alpha'_{\mathfrak{b}}(\mathfrak{x}_2(\mathbf{A}')) \in \mathfrak{K}$ for every $\mathfrak{b}$, it follows from (12′) that $\mathbf{A}'$ is invariant under all the automorphisms of $\bar{\mathfrak{K}}/\mathfrak{K}$, that is, $\mathbf{A}'$ is defined over $\mathfrak{K}$.

Corollary 3. *If* $\mathcal{V}$, $\mathcal{V}'$ *are two extensions of the same* ν*-covering* U, *then*

$$\Sigma(\mathcal{V}', \mathfrak{A}) = S\,\Sigma(\mathcal{V}, \mathfrak{A}),$$

where S *is an invariance class which depends only on* $\mathcal{V}$ *and* $\mathcal{V}'$ *but not on the point* $\mathfrak{A}$ *on* $\mathcal{D}$.

This follows at once from (14), where S is the class to which $\{A_{\mathfrak{b}}\}$ belongs.

The next two lemmas correspond to Lemmas 5 and 6 respectively.

Lemma 10. *Let* $\mathcal{V}$ *be a* ν^2*-covering which extends a* ν*-covering* U. *Let* $\mathfrak{A}$, $\mathfrak{b}$ *be points on* $\mathcal{D}$, $\mathcal{C}$ *respectively defined over* $\mathfrak{K}$, *and let* $\mathfrak{A} + \mathfrak{b}$ *be the point on* $\mathcal{D}$ *given by Lemma* 5. *Then*

$$\Sigma(\mathcal{V}, \mathfrak{A} + \mathfrak{b}) = \Sigma(\mathcal{V}, \mathfrak{A})\, \Sigma(\mathfrak{b}),$$

where $\Sigma(\mathfrak{b})$ *is the invariance class corresponding to* $\mathfrak{b}$ *under Lemma* 3 *and where the symbol* $\Sigma(\mathcal{V}, \mathfrak{A})$ *is defined in Lemma* 9, *Corollary* 2.

Let $\mathfrak{a}_2$, $\mathfrak{b}_1$, be points in the inverse image of $\mathfrak{A}$, $\mathfrak{b}$ under the respective maps $\mathfrak{x}_2 \to \mathfrak{X}$, $\mathfrak{x}_1 \to \mathfrak{x} = \nu\mathfrak{x}_1$. Then $\mathfrak{a}_2 + \mathfrak{b}_1$ is a point of the inverse image of $\mathfrak{A} + \mathfrak{b}$ under $\mathfrak{x}_2 \to \mathfrak{X}$. Hence, remembering the definitions of $\Sigma(\mathcal{V}, \mathfrak{A})$, $\Sigma(\mathfrak{b})$, what we have to prove is that

$$\frac{\alpha_{\mathfrak{b}}(\mathfrak{a}_2 + \mathfrak{b}_1)}{\alpha_{\mathfrak{b}}(\mathfrak{a}_2)\, \varphi_{\mathfrak{b}}(\mathfrak{b}_1)} \in \mathfrak{K}, \tag{19}$$

where $\varphi_{\mathfrak{b}}(\mathfrak{x}_1)$ is the function given by (8) of § 2.

Consider the function

$$\frac{\alpha_{\mathfrak{b}}(\mathfrak{x}_2 + \mathfrak{y}_1)}{\alpha_{\mathfrak{b}}(\mathfrak{x}_2)\, \varphi_{\mathfrak{b}}(\mathfrak{y}_1)} \tag{20}$$

of two independent generic points $\mathfrak{x}_2$, $\mathfrak{y}_1$ of $\mathcal{C}$. By (12) of § 2 and (12′) we see that (20) is unchanged by the substitution of $\mathfrak{x}_2 + \mathfrak{e}$ for $\mathfrak{x}_2$, where $\nu\mathfrak{e} = \mathfrak{o}$, and also by the substitution of $\mathfrak{y}_1 + \mathfrak{e}$ for $\mathfrak{y}_1$. Hence (20) is a function of the generic points

$$\mathfrak{x}_1 = \nu\mathfrak{x}_2, \quad \mathfrak{y} = \nu\mathfrak{y}_1. \tag{21}$$

Let $\mathfrak{X} = \mathfrak{X}(\mathfrak{x}_1)$ be the point on $\mathcal{D}$ corresponding to $\mathfrak{x}_1$ under the birational correspondence $\mathcal{C} \longleftrightarrow \mathcal{D}$ in (1). We have shown, then, that

$$\frac{\alpha_{\mathfrak{b}}(\mathfrak{x}_2 + \mathfrak{y}_1)}{\alpha_{\mathfrak{b}}(\mathfrak{x}_2)\,\varphi_{\mathfrak{b}}(\mathfrak{y}_1)} = \Theta_{\mathfrak{b}}(\mathfrak{X}, \mathfrak{y}) \in \bar{\mathfrak{K}}(\mathfrak{X}, \mathfrak{y}) \text{ (say).} \tag{22}$$

We want to show that, in fact, $\Theta_{\mathfrak{b}}$ is defined over $\mathfrak{K}$. Let $\mathbf{X}$ be a generic point over $\mathfrak{K}$ on $\mathcal{S}$ and let $\mathfrak{X}, \mathfrak{x}_2, \mathfrak{x}_1, \mathfrak{x}$ be determined by the diagram (1′), so $\mathfrak{X}$ is a generic point over $\mathfrak{K}$ on $\mathcal{D}$. Let $\mathbf{X}'$ be the point onto which $\mathfrak{x}_2 + \mathfrak{y}_1$ is mapped by the isomorphism $\mathcal{C} \longleftrightarrow \mathcal{S}$ of (1), where $\mathfrak{y}$, as before, is an independent generic point over $\mathfrak{K}$ on $\mathcal{C}$. Then, by (18),

$$\begin{cases} \alpha_{\mathfrak{b}}(\mathfrak{x}_2) \in \mathfrak{K}(\mathbf{X}) \\ \alpha_{\mathfrak{b}}(\mathfrak{x}_2 + \mathfrak{y}_1) \in \mathfrak{K}(\mathbf{X}'). \end{cases} \tag{23}$$

Further,

$$\varphi_{\mathfrak{b}}(\mathfrak{y}_1) \in \mathfrak{K}(\mathfrak{y}_1) \tag{24}$$

by definition ((8) of § 2). But now

$$\mathfrak{K}(\mathbf{X}') \subset \mathfrak{K}(\mathbf{X}, \mathfrak{y}_1) \tag{25}$$

by Lemma 5 with

$$\nu^2,\ \mathfrak{K}(\mathbf{X}),\ \mathbf{X}, \mathbf{X}', \mathfrak{y}_1,\ \mathcal{S}$$

instead of

$$\nu,\ \mathfrak{K},\ \mathfrak{A}',\ \mathfrak{X}'',\ \mathfrak{z},\ \mathcal{D}$$

respectively. By (22), (23) and (24) we have

$$\Theta_{\mathfrak{b}}(\mathfrak{X}, \mathfrak{y}) \in \mathfrak{K}(\mathbf{X}, \mathfrak{y}_1). \tag{26}$$

Since $\mathbf{X}$ and $\mathfrak{y}_1$ are independent generic points over $\mathfrak{K}$ we must have

$$\mathfrak{K}(\mathbf{X}, \mathfrak{y}_1) \cap \bar{\mathfrak{K}}(\mathfrak{X}, \mathfrak{y}) = \mathfrak{K}(\mathfrak{X}, \mathfrak{y}), \tag{27}$$

and so, by (22) and (25),

$$\Theta_{\mathfrak{b}}(\mathfrak{X}, \mathfrak{y}) \in \mathfrak{K}(\mathfrak{X}, \mathfrak{y}). \tag{28}$$

Finally, the specialisation $\mathfrak{X} \to \mathfrak{A}$, $\mathfrak{y} \to \mathfrak{b}$ in (28) shows that $\Theta_{\mathfrak{b}}(\mathfrak{A}, \mathfrak{b}) \in \mathfrak{K}$. This is precisely the statement (19) which we set out to prove.

Lemma 11. *Let U', U'' be two ν-coverings corresponding to invariance classes S' and S'' and let $\mathcal{D}' = \mathcal{D}(U')$, $\mathcal{D}'' = \mathcal{D}(U'')$. Suppose that they are extensible to ν^2-coverings $\mathcal{V}', \mathcal{V}''$ respectively with $\mathcal{S}' = \mathcal{S}(\mathcal{V}')$, $\mathcal{S}'' = \mathcal{S}(\mathcal{V}'')$. Then the ν-covering U corresponding to $S'S''$ is extensible. It is possible to choose the extension $\mathcal{V}$ of U so that*

$$\frac{\alpha'_{\mathfrak{b}}(\mathfrak{x}'_2)\,\alpha''_{\mathfrak{b}}(\mathfrak{x}''_2)}{\alpha_{\mathfrak{b}}(\mathfrak{x}'_2 + \mathfrak{x}''_2)} \in \mathfrak{K}(\mathfrak{X}', \mathfrak{X}''), \tag{29}$$

where $\alpha'_{\mathfrak{b}}, \alpha''_{\mathfrak{b}}, \alpha_{\mathfrak{b}}$ refer to $\mathcal{V}', \mathcal{V}'', \mathcal{V}$ respectively and $\mathfrak{X}', \mathfrak{X}''$ are the points on $\mathcal{D}', \mathcal{D}''$ corresponding to $\mathfrak{x}'_2, \mathfrak{x}''_2$ under the maps of $\mathcal{V}', \mathcal{V}''$ respectively.

We denote the objects in the diagrams corresponding to (1), (1′) for $\mathcal{V}', \mathcal{V}''$ by the corresponding letter in (1), (1′) with one or two dashes. For example $\mathbf{X}'$ is a generic point of $\mathcal{S}(\mathcal{V}') = \mathcal{S}'$ which is mapped onto a generic point $\mathfrak{X}'$ of $\mathcal{D}' = \mathcal{D}(U')$. The corresponding objects in the diagram (1), (1′) for the covering $\mathcal{V}$, which we are trying to construct, will be denoted as in (1), (1′), i. e. without dashes.

Let $\mathfrak{O}$ be a point on $\mathcal{D}$ defined over a finite normal algebraic extension $\mathfrak{L}$ of $\mathfrak{K}$ and let $\psi_{\mathfrak{b}}(\mathfrak{x}_2)$ be a function, defined over $\mathfrak{L}$, whose divisor is the inverse image of $(\mathfrak{O})^{-1}(\mathfrak{O}+\mathfrak{b})$ under $\mathfrak{x}_2 \to \mathfrak{X}$. We may suppose that all the maps of the ν^2-coverings $\mathcal{V}', \mathcal{V}''$ are defined over $\mathfrak{L}$. Then

$$\frac{\alpha_{\mathfrak{b}}'(\mathfrak{x}_2')\,\alpha_{\mathfrak{b}}''(\mathfrak{x}_2'')}{\psi_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')} = \Omega_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'') \in \mathfrak{L}(\mathfrak{X}', \mathfrak{X}''), \quad \text{(say)} \tag{30}$$

since it is unchanged by the substitutions

$$\mathfrak{x}_2' \to \mathfrak{x}_2' + \mathfrak{e} \qquad (\nu\mathfrak{e} = \mathfrak{o})$$

or

$$\mathfrak{x}_2'' \to \mathfrak{x}_2'' + \mathfrak{e} \qquad (\nu\mathfrak{e} = \mathfrak{o}).$$

Let ϱ be any automorphism of $\mathfrak{L}/\mathfrak{K}$. We extend ϱ to $\mathfrak{L}(\mathbf{X}', \mathbf{X}'') = \mathfrak{L}(\mathfrak{x}_2', \mathfrak{x}_2'')$ by putting

$$\varrho\mathbf{X}' = \mathbf{X}', \ \varrho\mathbf{X}'' = \mathbf{X}''. \tag{31}$$

Then

$$\begin{cases} \varrho\{\alpha_{\mathfrak{b}}'(\mathfrak{x}_2')\} = \alpha_{\mathfrak{b}}'(\mathfrak{x}_2') \\ \varrho\{\alpha_{\mathfrak{b}}''(\mathfrak{x}_2'')\} = \alpha_{\mathfrak{b}}''(\mathfrak{x}_2'') \end{cases} \tag{32}$$

by (18). Hence by (30)

$$\frac{\varrho\,\psi_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')}{\psi_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')} = \frac{\Omega_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'')}{\varrho\Omega_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'')}. \tag{33}$$

But now (either from (33), or by the argument preceding (2) of § 3),

$$\frac{\varrho\,\psi_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')}{\psi_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')} = \Psi_{\varrho,\mathfrak{b}}(\mathfrak{x}_1) \in \mathfrak{L}(\mathfrak{x}_1) \qquad \text{(say)}, \tag{34}$$

where $\mathfrak{x}_1 = \nu(\mathfrak{x}_2'+\mathfrak{x}_2'')$. From (33) it follows that, for fixed $\mathfrak{b}$, the $\Psi_{\varrho,\mathfrak{b}}(\mathfrak{x}_1)$ are a coboundary, that is

$$\Psi_{\varrho,\mathfrak{b}}(\mathfrak{x}_1)\,\varrho\Psi_{\varrho^{-1}\sigma,\mathfrak{b}}(\mathfrak{x}_1) = \Psi_{\sigma,\mathfrak{b}}(\mathfrak{x}_1) \tag{35}$$

for any two automorphs ϱ, σ of $\mathfrak{L}/\mathfrak{K}$. Hence, by Hilbert's Satz 90,

$$\Psi_{\varrho,\mathfrak{b}}(\mathfrak{x}_1) = \frac{\Lambda_{\mathfrak{b}}(\mathfrak{x}_1)}{\varrho\Lambda_{\mathfrak{b}}(\mathfrak{x}_1)} \tag{36}$$

for some $\Lambda_{\mathfrak{b}}(\mathfrak{x}_1) \in \mathfrak{K}(\mathfrak{x}_1)$. We put

$$\alpha_{\mathfrak{b}}(\mathfrak{x}_2) = \psi_{\mathfrak{b}}(\mathfrak{x}_2)\,\Lambda_{\mathfrak{b}}(\mathfrak{x}_1), \tag{37}$$

so

$$\varrho\alpha_{\mathfrak{b}}(\mathfrak{x}_2) = \alpha_{\mathfrak{b}}(\mathfrak{x}_2) \tag{38}$$

for all ϱ, and

$$\begin{aligned} \frac{\alpha_{\mathfrak{b}}'(\mathfrak{x}_2')\,\alpha_{\mathfrak{b}}''(\mathfrak{x}_2'')}{\alpha_{\mathfrak{b}}(\mathfrak{x}_2'+\mathfrak{x}_2'')} &= \frac{\Omega_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'')}{\Lambda_{\mathfrak{b}}(\mathfrak{x}_1)} \\ &= Y_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'') \qquad \text{(say)} \\ &\in \mathfrak{L}(\mathfrak{X}', \mathfrak{X}''), \end{aligned} \tag{39}$$

since $\mathfrak{x}_1 = \mathfrak{x}_1' + \mathfrak{x}_1'' \in \mathfrak{L}(\mathfrak{X}', \mathfrak{X}'')$. Now, by (32), (38) and (39) we have

$$\varrho Y_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'') = Y_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'')$$

for all automorphs ϱ of $\mathfrak{L}/\mathfrak{K}$. Since $\varrho\mathfrak{X}' = \mathfrak{X}'$, $\varrho\mathfrak{X}'' = \mathfrak{X}''$ by (31), it follows that $Y_{\mathfrak{b}}(\mathfrak{X}', \mathfrak{X}'') \in \mathfrak{K}(\mathfrak{X}', \mathfrak{X}'')$. The $\alpha_{\mathfrak{b}}(\mathfrak{x}_2)$ satisfy the condition (i) of Lemma 9 by (37) and the

definition of $\psi_b(\mathfrak{x}_2)$: they satisfy (ii) of Lemma (9) by (39), since the $\alpha_b'(\mathfrak{x}_2')$, $\alpha_b''(\mathfrak{x}_2'')$ do this hypothesis, and since $Y_b(\mathfrak{X}', \mathfrak{X}'') \in \mathfrak{K}(\mathfrak{X}', \mathfrak{X}'')$. This completes the proof of the Lemma.

We shall need only the following two corollaries.

Corollary 1. *Let $\mathcal{V}$ be the extension of U constructed in Lemma 11. Then the map $(\mathfrak{x}_2', \mathfrak{x}_2'') \to \mathfrak{x}_2 = \mathfrak{x}_2' + \mathfrak{x}_2''$ induces a map $(\mathbf{X}', \mathbf{X}'') \to \mathbf{X}$ defined over $\mathfrak{K}$, where $\mathbf{X}', \mathbf{X}'', \mathbf{X}$ are generic points of $\mathcal{S}', \mathcal{S}'', \mathcal{S}$ respectively.*

This was shown in the course of the proof.

Corollary 2. *Let $\mathfrak{A}', \mathfrak{A}''$ be points on $\mathcal{D}', \mathcal{D}''$ respectively defined over $\mathfrak{K}$ and let $\mathfrak{A} = \mathfrak{A}' + \mathfrak{A}''$ be on $\mathcal{D}$ in the notation of the note after Lemma 6. Let $\mathcal{V}', \mathcal{V}'', \mathcal{V}$ be the respective extensions of U', U'', U given by Lemma 11. Then, in the notation of Lemma 9, Corollary 2,*

$$\Sigma(\mathcal{V}, \mathfrak{A}) = \Sigma(\mathcal{V}', \mathfrak{A}')\, \Sigma(\mathcal{V}'', \mathfrak{A}'').$$

The truth of the Corollary follows at once from the specialization $\mathfrak{x}_2' \to \mathfrak{a}_2'$, $\mathfrak{x}_2'' \to \mathfrak{a}_2''$ in (29), in an obvious notation.

§ 4. A mutual invariant of two invariance classes.

In this section we continue to assume that $\mathfrak{K}$ is any field of characteristic 0 for which conditions (I), (II) of Theorem I hold. This section is devoted to the proof of the following result, which is the key to all that follows. If $\mathfrak{L}$ is any field we denote by $\mathfrak{L}^*$ the multiplicative group of non-zero elements of $\mathfrak{L}$.

If $\mathfrak{L}$ is a finite normal extension of $\mathfrak{K}$ we denote by ${}^jH(\mathfrak{L}/\mathfrak{K})$ the j-th cohomology group[22]) of the module $\mathfrak{L}^*$ with respect to the Galois group of $\mathfrak{L}/\mathfrak{K}$. We denote by ${}^jH(\bar{\mathfrak{K}}/\mathfrak{K})$ the continuous homology group with the usual topology on the Galois group of $\bar{\mathfrak{K}}/\mathfrak{K}$, where $\bar{\mathfrak{K}}$ is, as before, the algebraic closure of $\mathfrak{K}$.

In this section we prove

Theorem II. *There is a natural mapping*

$$(S', S'') \to h(S', S'')$$

of pairs (S', S'') of invariance classes into ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ with the following properties:

(i) $h(S'S''', S'') = h(S', S'')\, h(S''', S'')$,

(ii) $h(S', S''S''') = h(S', S'')\, h(S', S''')$;

(iii) $h(S', S'') = 1$ *if there are points defined over $\mathfrak{K}$ on each of the curves $\mathcal{D}(S'), \mathcal{D}(S'')$.*

It will be convenient to deal with normal commutative semi-simple algebras[23]) $\mathfrak{M}$ over $\mathfrak{K}$. Such algebras arise naturally by adjoining roots of reducible equations to $\mathfrak{K}$: another example occurs when a field is completed with respect to a valuation: if $\mathfrak{L}$ is a finite algebraic extension of an algebraic number-field $\mathfrak{K}$ and $\mathfrak{K}_\mathfrak{p}$ is the closure of $\mathfrak{K}$ with respect to a valuation $\mathfrak{p}$, the tensor product $\mathfrak{L}_\mathfrak{p}$ of $\mathfrak{L}$ and $\mathfrak{K}_\mathfrak{p}$ over $\mathfrak{K}$ is an algebra over $\mathfrak{K}$, but it is not, in general a field. We shall be interested in the case when the algebra $\mathfrak{M}$ is normal, i. e. when there is a group Γ of automorphisms of $\mathfrak{M}/\mathfrak{K}$, the fixed ring of Γ

[22]) In my use of the theory of the homology of group-modules I have largely followed Chevalley (3).

[23]) Cf. Hasse (4), (5).

being precisely $\mathfrak{K}$. Since $\mathfrak{K}$ is of characteristic 0, it is then familiar that $\mathfrak{M}$ is the direct sum of a finite number of isomorphic images of a normal field extension of $\mathfrak{K}$. If $\mathfrak{L}$ is one of these copies, with isomorphism-group Δ, then

$$\mathfrak{M} = \sum_i \tau_i \mathfrak{L} \quad \text{(direct)},$$

where $\Gamma = \sum_i \tau_i \Delta$. If $\mathfrak{M}^*$ denotes the multiplicative group of invertible elements of $\mathfrak{M}$, there is then a natural isomorphism[24])

$$(1) \qquad {}^jH(\Gamma, \mathfrak{M}^*) \cong {}^jH(\Delta, \mathfrak{L}^*) = {}^jH(\mathfrak{L}/\mathfrak{K}).$$

If $j = 2$, the case which will most concern us[25]), the group ${}^2H(\mathfrak{L}/\mathfrak{K})$ can be identified with a subgroup of ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ (cf. Artin (**1**) or Chevalley (**3**)). Consequently we shall also identify

$${}^2H(\Gamma, \mathfrak{M}^*) = {}^2H(\mathfrak{M}/\mathfrak{K})$$

with this subgroup.

An invariant system $\{A_\mathfrak{b}\}$ defines a normal algebra $\mathfrak{M}\{A_\mathfrak{b}\}$ of degree[26]) N over $\mathfrak{K}$ with basis $\tilde{A}_\mathfrak{b}$ (say) and multiplication scheme

$$(2) \qquad \tilde{A}_{\mathfrak{b}_1}\tilde{A}_{\mathfrak{b}_2} = \left(\frac{A_{\mathfrak{b}_1}A_{\mathfrak{b}_2}}{A_{\mathfrak{b}_1+\mathfrak{b}_2}}\right)\tilde{A}_{\mathfrak{b}_1+\mathfrak{b}_2},$$

where, by the definition of an invariant system,

$$(3) \qquad \left(\frac{A_{\mathfrak{b}_1}A_{\mathfrak{b}_2}}{A_{\mathfrak{b}_1+\mathfrak{b}_2}}\right) \in \mathfrak{K}.$$

If $\nu\mathfrak{e} = \mathfrak{o}$, we denote by $\sigma_\mathfrak{e}$ the automorphism of $\mathfrak{M}\{A_\mathfrak{b}\}/\mathfrak{K}$ for which

$$(4) \qquad \sigma_\mathfrak{e}\tilde{A}_\mathfrak{b} = \{\chi(\mathfrak{b}, \mathfrak{e})\}^{-1}\tilde{A}_\mathfrak{b}.$$

It follows easily from (2) that the $\sigma_\mathfrak{e}$ are the complete group of automorphisms of $\mathfrak{M}\{A_\mathfrak{b}\}/\mathfrak{K}$. Two invariant systems in the same invariance class S clearly give the same algebra, so we may write $\mathfrak{M}(S)$ instead of $\mathfrak{M}\{A_\mathfrak{b}\}$.

If now S', S'' are two invariance classes and if

$$(5) \qquad \{A_\mathfrak{b}''\} \in S'',$$

we denote by $h^*(S', S'')$ the element of ${}^2H(\mathfrak{M}(S')/\mathfrak{K})$ with representative cocycle[27])

$$(6) \qquad (\sigma_\mathfrak{b}', \sigma_\mathfrak{e}', A_\mathfrak{e}''A_{\mathfrak{b}-\mathfrak{e}}''/A_\mathfrak{b}'')$$

(i. e. the cocycle makes the element $A_\mathfrak{e}''A_{\mathfrak{b}-\mathfrak{e}}''/A_\mathfrak{b}'' \in \mathfrak{K}^*$ correspond to the ordered pair of elements $\sigma_\mathfrak{b}', \sigma_\mathfrak{e}$ of the galois group of $\mathfrak{M}(S')/\mathfrak{K}$). Clearly (6) is a cocycle and the sub-

---

[24]) See Appendix.

[25]) Note however that the case $j = 1$ shows that ${}^1H(\Gamma, \mathfrak{M}^*)$ is trivial: that is, Noether's generalisation of Hilbert's Satz 90 holds for commutative algebras.

[26]) N is the number of points in the kernel of the mapping $\mathfrak{x} \to \nu\mathfrak{x}$.

[27]) As several different notations are in vogue for the homology groups of modules with respect to a finite group of automorphisms, we state ours explicitly here. If A is a Γ-module, written additively, where Γ is a finite group with unit ε, the 1-cocycles, 2-cocycles have coefficients a_γ, $a_{\gamma,\delta} \in A$ respectively, where

$$a_\varepsilon = 0, \ \gamma a_{\gamma^{-1}\delta} = a_\delta - a_\gamma \quad (\text{all } \gamma, \delta \in \Gamma)$$

$$a_{\varepsilon,\gamma} = a_{\gamma,\gamma} = 0, \ \beta a_{\beta^{-1}\gamma,\beta^{-1}\delta} = a_{\gamma,\delta} + a_{\beta,\gamma} - a_{\beta,\delta}.$$

The 1-coboundaries, 2-coboundaries are respectively the cocycles of the shape

$$a_\gamma = \gamma b - b, \quad a_{\gamma,\delta} = \gamma b_{\gamma^{-1}\delta} + b_\gamma - b_\delta$$

where

$$b \in A, \ b_\gamma \in A, \ b_\varepsilon = 0.$$

stitution of another invariant system in S'' for $\{A''_{\mathfrak{b}}\}$ merely multiplies (6) by a coboundary. Hence $h^*(S', S'')$ is well-defined. Thus the injection mapping

$$^2H(\mathfrak{M}(S')/\mathfrak{K}) \to {}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$$

takes $h^*(S', S'')$ into a uniquely defined element $h(S', S'') \in {}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$.

It remains to verify the statements (i), (ii), (iii) of Theorem II.

(ii) follows at once from the expression (6) for a representative cocycle.

(i) is also immediate, on lifting the representative cocycles to cocycles of $^2H(\mathfrak{M}(S')\,\mathfrak{M}(S''')/\mathfrak{K})$ where $\mathfrak{M}(S')\,\mathfrak{M}(S''')$ is the tensor product of $\mathfrak{M}(S')$ and $\mathfrak{M}(S''')$ over $\mathfrak{K}$.

It thus remains only to prove (iii). Let $\mathfrak{A}', \mathfrak{A}''$ be points of $\mathcal{D}(S')$, $\mathcal{D}(S'')$ respectively defined over $\mathfrak{K}$ and let $\mathfrak{a}', \mathfrak{a}''$ be the points of $\mathcal{C}$ defined over $\mathfrak{K}$ onto which they are canonically mapped. Let $\mathfrak{a}'_1, \mathfrak{a}''_1, \mathfrak{a}'_2, \mathfrak{a}''_2$ be points of $\mathcal{C}$ defined over $\bar{\mathfrak{K}}$ for which

$$\begin{cases} \nu\mathfrak{a}'_2 = \mathfrak{a}'_1, & \nu\mathfrak{a}'_1 = \mathfrak{a}', \\ \nu\mathfrak{a}''_2 = \mathfrak{a}''_1, & \nu\mathfrak{a}''_1 = \mathfrak{a}''. \end{cases} \tag{7}$$

By Lemma 1 there is a function $g(\mathfrak{x})$ of the generic point $\mathfrak{x}$ of $\mathcal{C}$ defined over $\mathfrak{K}$ such that the function

$$\varphi_{\mathfrak{b}}(\mathfrak{x}) = \frac{\Phi_{\mathfrak{b}}(\mathfrak{x})\, g(\nu\mathfrak{x})}{g(\nu\mathfrak{x} - \mathfrak{b})} \tag{8}$$

has the properties:

$$\varphi_{\mathfrak{b}}(\mathfrak{x} + \mathfrak{e}) = \chi(\mathfrak{b}, \mathfrak{e})\, \varphi_{\mathfrak{b}}(\mathfrak{x}) \tag{I}$$

for all $\mathfrak{e}$ with $\nu\mathfrak{e} = \mathfrak{o}$.

(II) No solution of

$$\nu^2\mathfrak{x} = \nu\mathfrak{a}'', \ \nu\mathfrak{a}', \ \nu(\mathfrak{a}' + \mathfrak{a}'')$$

is a zero or pole of any $\varphi_{\mathfrak{b}}(\mathfrak{x})$ and

$$\varphi_{\mathfrak{b}}(\mathfrak{x}) \in \mathfrak{K}(\mathfrak{x}). \tag{III}$$

Then $\{A'_{\mathfrak{b}}\} = \{\varphi_{\mathfrak{b}}(\mathfrak{a}'_1)\}$ is an invariant system belonging to the invariance class S' by Lemma 3; and the automorphisms $\sigma'_{\mathfrak{e}}$ are given by the substitutions

$$\sigma'_{\mathfrak{e}}\colon \mathfrak{a}'_1 \to \mathfrak{a}'_1 - \mathfrak{e}. \tag{8'}$$

Similarly for S'', $\mathfrak{a}''_1$ and $\sigma''_{\mathfrak{e}}$.

We now consider the functions

$$\vartheta_{\mathfrak{b}}(\mathfrak{x}_1, \mathfrak{y}) = \frac{\varphi_{\mathfrak{b}}(\mathfrak{x}_2)\, \varphi_{\mathfrak{b}}(\mathfrak{y}_1)}{\varphi_{\mathfrak{b}}(\mathfrak{x}_2 + \mathfrak{y}_1)} \in \mathfrak{K}(\mathfrak{x}_1, \mathfrak{y}) \tag{9}$$

of two generic points $\mathfrak{x}_1 = \nu\mathfrak{x}_2$, $\mathfrak{y} = \nu\mathfrak{y}_1$ of $\mathcal{C}$, which were defined in Lemma 2. A comparison of zeros and poles shows that

$$\frac{\Phi_{\mathfrak{e}}(\mathfrak{x}_2)\, \Phi_{\mathfrak{b}-\mathfrak{e}}(\mathfrak{x}_2 - \mathfrak{f})}{\Phi_{\mathfrak{b}}(\mathfrak{x}_2)} = \lambda_{\mathfrak{b},\mathfrak{e}} \in \bar{\mathfrak{K}} \tag{10}$$

is independent of $\mathfrak{x}_2$, where $\mathfrak{f}$ is any fixed solution of $\nu\mathfrak{f} = \mathfrak{e}$. Hence, by (8) and (9),

$$\frac{\vartheta_{\mathfrak{e}}(\mathfrak{x}_1, \mathfrak{y})\, \vartheta_{\mathfrak{b}-\mathfrak{e}}(\mathfrak{x}_1 - \mathfrak{e}, \mathfrak{y})}{\vartheta_{\mathfrak{b}}(\mathfrak{x}_1, \mathfrak{y})} = \frac{\varphi_{\mathfrak{e}}(\mathfrak{y}_1)\varphi_{\mathfrak{b}-\mathfrak{e}}(\mathfrak{y}_1)}{\varphi_{\mathfrak{b}}(\mathfrak{y}_1)}. \tag{11}$$

We now make the specialisation $\mathfrak{x}_2 \to \mathfrak{a}_2'$, $\mathfrak{y}_1 \to \mathfrak{a}_1''$. By (III) in the construction of $\varphi_\mathfrak{b}(\mathfrak{x})$, none of the $\vartheta_\mathfrak{b}(\mathfrak{a}_1' - \mathfrak{c}, \mathfrak{a}'')$ and $\varphi_\mathfrak{b}(\mathfrak{a}_1'')$ is 0 or ∞. Thus, by (8′), the specialisation of (11) gives $\dfrac{B_\mathfrak{c}\sigma_\mathfrak{c}' B_{\mathfrak{b}-\mathfrak{c}}}{B_\mathfrak{b}} = \dfrac{A_\mathfrak{c}'' A_{\mathfrak{b}-\mathfrak{c}}}{A_\mathfrak{b}''}$ with $B_\mathfrak{b} = \vartheta_\mathfrak{b}(\mathfrak{a}_1, \mathfrak{b})$ and $A_\mathfrak{b}'' = \varphi_\mathfrak{b}(\mathfrak{a}_1'')$, that is expresses the representative cocycle (6), with $A_\mathfrak{b}'' = \varphi_\mathfrak{b}(\mathfrak{a}_1'')$ as a coboundary: and so $h(S', S'') = 1$, as required.

Corollary 1. *There is a natural mapping of the pair consisting of an invariance class S and an extension $\mathcal{V}$ of $U(S)$ into an element $h(S, \mathcal{V}) \in {}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ with the following properties:*

(i) *If $\mathfrak{A}$ is any point of $\mathcal{D}(S)$ defined over $\mathfrak{K}$, then*

$$h(S, \Sigma(\mathcal{V}, \mathfrak{A})) = h(S, \mathcal{V}),$$

where $\Sigma(\mathcal{V}, \mathfrak{A})$ is the invariance system given by Lemma 9, Corollary 2.

(ii) *Suppose that $\mathfrak{K}$ is a subfield of a field $\mathfrak{K}_1$, and that $\bar{\mathfrak{K}} < \bar{\mathfrak{K}}_1$. Let $h_1(S, \mathcal{V})$ be constructed with respect to $\mathfrak{K}_1$ in the same way as $h(S, \mathcal{V})$ is constructed with respect to $\mathfrak{K}$. Then $h_1(S, \mathcal{V})$ is the image of $h(S, \mathcal{V})$ in the natural mapping*[28])

$${}^2H(\bar{\mathfrak{K}}/\mathfrak{K}) \to {}^2H(\bar{\mathfrak{K}}_1/\mathfrak{K}_1).$$

Let $\mathfrak{L}$ be one of the normal field extensions of $\mathfrak{K}$ which occur as direct summands in the normal algebra $\mathfrak{M}(S)$. Then the galois group of $\mathfrak{L}/\mathfrak{K}$, being a subgroup of the galois group of $\mathfrak{M}(S)/\mathfrak{K}$, is the set of

$$\sigma_\mathfrak{b} \qquad (\mathfrak{b} \in \Delta_0) \tag{12}$$

in the notation of (4) and (6), where Δ_0 is a subgroup of the additive group of the $\mathfrak{b}$. Further, the cocycle (6) corresponds in the isomorphism (1) between ${}^2H(\mathfrak{M}(S)/\mathfrak{K})$ and ${}^2H(\mathfrak{L}/\mathfrak{K})$ to the cocycle of ${}^2H(\mathfrak{L}/\mathfrak{K})$ which has the same form as (6), except that $\mathfrak{b}$, $\mathfrak{c}$ are restricted to be in Δ_0.

Let $\mathbf{X}$, $\mathfrak{X}$, $\mathfrak{x}_2$, $\mathfrak{x}_1$, $\mathfrak{x}$ be a set of generic points connected by the usual maps (1′). We may extend the isomorphisms (12) in some fixed way to $\bar{\mathfrak{K}}$ and then to $\bar{\mathfrak{K}}(\mathbf{X}) = \bar{\mathfrak{K}}(\mathfrak{x}_2)$ by putting

$$\sigma_\mathfrak{b}\mathbf{X} = \mathbf{X} \qquad (\mathfrak{b} \in \Delta_0) \tag{13}$$

and so

$$\sigma_\mathfrak{b}\mathfrak{X} = \mathfrak{X}. \tag{14}$$

Then

$$\sigma_\mathfrak{b}\mathfrak{x}_1 = \mathfrak{x}_1 - \mathfrak{b} \tag{15}$$

by comparison of (4) and (12) of this § with (16) of § 2.

We denote by $\mathfrak{O}$ the image of $\mathfrak{o}$ in the map $\mathfrak{x}_1 \to \mathfrak{X}$, so $\mathfrak{O}$ is defined over $\mathfrak{L}$ because the map is. Let

$$\Pi_\mathfrak{b}(\mathfrak{X}) \in \mathfrak{L}(\mathfrak{X}) \tag{16}$$

have the divisor $\mathfrak{D}(\mathfrak{b})\,\mathfrak{O}(\mathfrak{O} + \mathfrak{b})^{-1}$, as is possible by the definition of $\mathfrak{D}(\mathfrak{b})$. Comparison of zeros and poles shows that

$$\frac{\alpha_\mathfrak{b}(\mathfrak{x}_2)}{\Phi_\mathfrak{b}(\mathfrak{x}_2)\,\Pi_\mathfrak{b}(\mathfrak{X})} = \mu_\mathfrak{b} \in \bar{\mathfrak{K}}, \tag{17}$$

---

[28]) This mapping is defined as follows. Since in ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ we are concerned only with continuous cocycles, every element of ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ can be represented by a cocycle $\{c_{\varrho,\sigma}\}$, where $c_{\varrho,\sigma} \in \mathfrak{K}^*$ and ϱ, σ run through the automorphisms of a finite normal extension $\mathfrak{L}/\mathfrak{K}$. The corresponding element of ${}^2H(\bar{\mathfrak{K}}_1/\mathfrak{K}_1)$ is then represented by the cocycle $\{c_{\varrho,\sigma}\}$, where now ϱ, σ are restricted to leave $\mathfrak{K}_1$ elementwise invariant, i. e. they are automorphisms of $\mathfrak{K}_1\mathfrak{L}/\mathfrak{K}_1$.

where $\mathfrak{X} = \mathfrak{X}(\mathfrak{x}_2)$. Now

$$\alpha_{\mathfrak{b}}(\mathfrak{x}_2) \in \mathfrak{K}(\mathbf{X}) \tag{18}$$

by (18) of § 3, and so

$$\frac{\alpha_{\mathfrak{b}}(\mathfrak{x}_2)\,\alpha_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{x}_2)}{\alpha_{\mathfrak{e}}(\mathfrak{x}_2)} = \frac{\alpha_{\mathfrak{b}}(\mathfrak{x}_2)\,\sigma_{\mathfrak{b}}\{\alpha_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{x}_2)\}}{\alpha_{\mathfrak{e}}(\mathfrak{x}_2)} = \gamma_{\mathfrak{b},\mathfrak{e}}\,\frac{\Pi_{\mathfrak{b}}(\mathfrak{X})\,\sigma_{\mathfrak{b}}\{\Pi_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{X})\}}{\Pi_{\mathfrak{e}}(\mathfrak{X})} \tag{19}$$

where, by (10) and (15),

$$\gamma_{\mathfrak{b},\mathfrak{e}} = \frac{\lambda_{\mathfrak{e},\mathfrak{b}}\,\mu_{\mathfrak{b}}\,\sigma_{\mathfrak{b}}(\mu_{\mathfrak{e}-\mathfrak{b}})}{\mu_{\mathfrak{e}}} \in \bar{\mathfrak{K}}.$$

Comparison of (16), (18) and (19) shows that, in fact,

$$\gamma_{\mathfrak{b},\mathfrak{e}} \in \mathfrak{L}(\mathbf{X}) \cap \bar{\mathfrak{K}} = \mathfrak{L}.$$

By construction,

$$(\sigma_{\mathfrak{b}}, \sigma_{\mathfrak{e}}, \gamma_{\mathfrak{b},\mathfrak{e}}) \tag{20}$$

is a cocycle for $\mathfrak{L}/\mathfrak{K}$. We wish to show that the corresponding element $h(S, \mathcal{V})$ of ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ has the properties enunciated. The property (ii) is trivially fulfilled, since replacement of $\mathfrak{K}$ by an overfield $\mathfrak{K}_1$ merely replaces the group Δ_0 in (12) by a subgroup (cf. footnote [28])).

We now verify (i). A representative cocycle of $h(S, \Sigma(\mathcal{V}, \mathfrak{A}))$ is, by definition,

$$\big(\sigma_{\mathfrak{b}}, \sigma_{\mathfrak{e}}, \alpha_{\mathfrak{b}}(\mathfrak{a}_2)\,\alpha_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{a}_2)/\alpha_{\mathfrak{e}}(\mathfrak{a}_2)\big), \tag{20'}$$

where $\mathfrak{a}_2$ is a point of the inverse image of $\mathfrak{A}$ under $\mathfrak{x}_2 \to \mathfrak{X}$, and, as in Lemma 9, Corollary 2, the functions $\alpha_{\mathfrak{b}}(\mathfrak{x}_2)$ are chosen so that

$$\alpha_{\mathfrak{b}}(\mathfrak{a}_2) \neq 0,\ \infty. \tag{21}$$

Provided that none of the $\Pi_{\mathfrak{b}}(\mathfrak{A})$ is 0 or ∞ we may specialize $\mathfrak{x}_2$, $\mathfrak{X}$ in (19) to $\mathfrak{a}_2$, $\mathfrak{A}$ respectively, since (14) ensures that $\sigma_{\mathfrak{b}}\Pi_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{X})$ specializes into $\sigma_{\mathfrak{b}}\Pi_{\mathfrak{e}-\mathfrak{b}}(\mathfrak{A})$. The specialized form shows that (20) and (20′) differ only by a cocycle, which proves the statement (i) in this case. To cope with the exceptions we note that, by Lemma 1, we may choose $g(\mathfrak{x})$ in (8) so that

$$\varphi_{\mathfrak{b}}(\mathfrak{a}_2) \neq 0,\ \infty. \tag{21'}$$

Put

$$\pi_{\mathfrak{b}}(\mathfrak{x}) = \Pi_{\mathfrak{b}}(\mathfrak{x})\,\frac{g(\nu\mathfrak{x} - \mathfrak{b})}{g(\mathfrak{x})},$$

so that (17) continues to hold when Φ, Π are replaced by φ, π respectively, and hence

$$\varphi_{\mathfrak{b}}(\mathfrak{a}_2) \neq 0,\ \infty \tag{21''}$$

by (21) and (21′). Since $g(\mathfrak{x}_1) \in \mathfrak{K}(\mathfrak{x}_1)$, by definition, it follows from (15) that (19) continues to hold when π is substituted for Π. We can now make the specialization in the new form of (19) without trouble. This completes the proof of the corollary.

The following result is included only for interest. We do not use it later.

Corollary 2. *If $\nu = n$ is a rational integer, then*

$$h(S', S'') = h(S'', S'). \tag{22'}$$

If ν is not divisible by any rational integer except 1, then

$$h(S', S'')\,h(S'', S') = 1. \tag{22'}$$

The case treated in detail in the first paper of this series (2), namely $\nu = \sqrt{-3}$, is of the second type, and there $h(S', S'')$ was expressed as a norm residue symbol.

Write $\mathfrak{M}' = \mathfrak{M}(S')$, $\mathfrak{M}'' = \mathfrak{M}(S'')$ and let $\mathfrak{M}'\mathfrak{M}''$ be the tensor product of $\mathfrak{M}'$ $\mathfrak{M}''$ over $\mathfrak{K}$, i. e. the algebra of degree N^2 over $\mathfrak{K}$ with basis $\tilde{A}'_{\mathfrak{d}}\tilde{A}''_{\mathfrak{e}}$ ($\nu\mathfrak{d} = \nu\mathfrak{e} = \mathfrak{o}$). Let $\hat{h}(S' S'')$, $\hat{h}(S'', S')$ be the images of $h^*(S', S'')$, $h^*(S'', S')$ under the respective lift mappings

$$^2H(\mathfrak{M}'/\mathfrak{K}) \to {}^2H(\mathfrak{M}'\mathfrak{M}''/\mathfrak{K}),$$
$$^2H(\mathfrak{M}''/\mathfrak{K}) \to {}^2H(\mathfrak{M}'\mathfrak{M}''/\mathfrak{K}),$$

where

$$h^*(S', S'') \in {}^2H(\mathfrak{M}'/\mathfrak{K}),$$
$$h^*(S'', S') \in {}^2H(\mathfrak{M}''/\mathfrak{K})$$

are the cocycles constructed in the proof of Theorem II and map canonically into $h(S', S'')$, $h(S'', S)$. It will clearly be enough to show that

$$\hat{h}(S', S'') = \hat{h}(S'', S'),$$
$$\hat{h}(S', S'')\,\hat{h}(S'', S') = 1,$$

respectively. By (6) a representative cocycle for $\hat{h}(S', S'')$ is[29])

$$(\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1},\ \sigma'_{\mathfrak{d}_2}\sigma''_{\mathfrak{e}_2},\ A''_{\mathfrak{d}_1}A''_{\mathfrak{d}_2-\mathfrak{d}_1}/A_{\mathfrak{d}_2}). \tag{23}$$

Now the general coboundary for $\mathfrak{M}'\mathfrak{M}''/\mathfrak{K}$ is given by

$$\left(\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1},\ \sigma'_{\mathfrak{d}_2}\sigma''_{\mathfrak{e}_2},\ \frac{Q(\mathfrak{d}_1,\mathfrak{e}_1)\,\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1}Q(\mathfrak{d}_2-\mathfrak{d}_1,\mathfrak{e}_2-\mathfrak{e}_1)}{Q(\mathfrak{d}_2,\mathfrak{e}_2)}\right), \tag{24}$$

where $Q(\mathfrak{d}, \mathfrak{e})$ are any[30]) N^2 elements of $(\mathfrak{M}'\mathfrak{M}'')^*$ with $Q(\mathfrak{o}, \mathfrak{o}) = 1$. If, in particular,

$$Q(\mathfrak{d}, \mathfrak{e}) = \tilde{A}''_{\mathfrak{d}}, \tag{25}$$

we see by (4) that

$$\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1}Q(\mathfrak{d}_2-\mathfrak{d}_1,\ \mathfrak{e}_2-\mathfrak{e}_1) = \{\chi(\mathfrak{d}_2-\mathfrak{d}_1, \mathfrak{e}_1)\}^{-1}\tilde{A}''_{\mathfrak{d}_2-\mathfrak{d}_1}.$$

Hence, on dividing (23) by the coboundary corresponding to (25), we see that

$$(\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1},\ \sigma'_{\mathfrak{d}_2}\sigma''_{\mathfrak{e}_2},\ \{\chi(\mathfrak{d}_2-\mathfrak{d}_1, \mathfrak{e}_1)\}^{-1}) \tag{26}$$

is a representative cocycle for $\hat{h}(S', S'')$. Similarly

$$(\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1},\ \sigma'_{\mathfrak{d}_2}\sigma''_{\mathfrak{e}_2},\ \{\chi(\mathfrak{e}_2-\mathfrak{e}_1), \mathfrak{d}_1\}^{-1}) \tag{27}$$

is a representative cocycle for $\hat{h}(S'', S')$.

Now put

$$Q(\mathfrak{d}, \mathfrak{e}) = \chi(\mathfrak{d}, \mathfrak{e})$$

in (24). Since $\chi(\mathfrak{d}, \mathfrak{e}) \in \mathfrak{K}$ by (7) of § 2, we have

$$\frac{Q(\mathfrak{d}_1, \mathfrak{e}_1)\,\sigma'_{\mathfrak{d}_1}\sigma''_{\mathfrak{e}_1}Q(\mathfrak{d}_2-\mathfrak{d}_1, \mathfrak{e}_2-\mathfrak{e}_1)}{Q(\mathfrak{d}_2, \mathfrak{e}_2)} = \frac{\chi(\mathfrak{d}_1, \mathfrak{e}_1)\,\chi(\mathfrak{d}_2-\mathfrak{d}_1, \mathfrak{e}_2-\mathfrak{e}_1)}{\chi(\mathfrak{d}_2, \mathfrak{e}_2)}.$$

---

[29]) The automorphisms of $\mathfrak{M}'\mathfrak{M}''/\mathfrak{K}$ are the $\sigma'_{\mathfrak{d}}\sigma''_{\mathfrak{e}}$, where $\nu\mathfrak{d} = \nu\mathfrak{e} = \mathfrak{o}$, and $\sigma'_{\mathfrak{d}}, \sigma''_{\mathfrak{e}}$ are the automorphisms of $\mathfrak{M}'/\mathfrak{K}$, $\mathfrak{M}''/\mathfrak{K}$ respectively given by (4).

[30]) N is the order of the group of the $\mathfrak{d}$, so $N = |\nu|^2$.

On writing $\mathfrak{d}_2 = \mathfrak{d}_1 + (\mathfrak{d}_2 - \mathfrak{d}_1)$, $\mathfrak{e}_2 = \mathfrak{e}_1 + (\mathfrak{e}_2 - \mathfrak{e}_1)$ and using the multiplicative nature of χ, we see that this is equal to

$$\chi^{-1}(\mathfrak{d}_2 - \mathfrak{d}_1, \mathfrak{e}_1)\, \chi^{-1}(\mathfrak{d}_1, \mathfrak{e}_2 - \mathfrak{e}_1). \tag{28}$$

We must now distinguish the two cases of the enunciation. If $\nu = n$, it is well-known that

$$\chi(\mathfrak{d}, \mathfrak{d}) = 1 \qquad (\text{all } \mathfrak{d}),$$

so

$$\chi(\mathfrak{d}, \mathfrak{e})\, \chi(\mathfrak{e}, \mathfrak{d}) = 1.$$

Hence the expression (28) is

$$\chi^{-1}(\mathfrak{d}_2 - \mathfrak{d}_1, \mathfrak{e}_1)\, \chi(\mathfrak{e}_2 - \mathfrak{e}_1, \mathfrak{d}_1),$$

which shows that the two cocycles (26), (27) differ only by a coboundary, i. e. $\hat{h}(S', S'') = h(S'', S')$, as required.

If ν is not divisible by any rational integer $\neq 1$, the group of $\mathfrak{d}$ is cyclic, say with generator $\mathfrak{d}_0$. If now $\mathfrak{d} = r\mathfrak{d}_0$, $\mathfrak{e} = s\mathfrak{d}_0$ are any two solutions of $\nu\mathfrak{d} = \nu\mathfrak{e} = \mathfrak{o}$ we have

$$\chi(\mathfrak{d}, \mathfrak{e}) = \chi(r\mathfrak{d}_0, s\mathfrak{d}_0) = \{\chi(\mathfrak{d}_0, \mathfrak{d}_0)\}^{rs} = \chi(\mathfrak{e}, \mathfrak{d}).$$

Hence (28) in this case is

$$\chi^{-1}(\mathfrak{d}_2 - \mathfrak{d}_1, \mathfrak{e}_1)\, \chi^{-1}(\mathfrak{e}_2 - \mathfrak{e}_1, \mathfrak{d}_1),$$

so the product of (26) and (27) is a cocycle, i. e.

$$\hat{h}(S', S'')\, \hat{h}(S'', S') = 1,$$

as required.

§ 5. The local reciprocity.

In this section we treat the case when $\mathfrak{K} = \mathfrak{K}_\mathfrak{p}$ is the completion of a finite algebraic extension of the rationals with respect to a non-archimedean valuation $\mathfrak{p}$. We denote by $M = M_\mathfrak{p}$ the set of invariance systems for which there is a point defined over $\mathfrak{K}$ on $\mathcal{D}(S)$. By Weil's isomorphism (Lemma 3), the group M is isomorphic to $\mathfrak{G}/\nu\mathfrak{G}$, where $\mathfrak{G}$ is the group of points on C defined over $\mathfrak{K}$. We shall assume, as always, that conditions (I) and (II) of Theorem I are satisfied and will prove

Theorem III. *Let S' be an invariance class. Each of the following three statements implies the other two:*

(i) $S' \in M$.

(ii) $h(S', S'') = 1$ *for all* $S'' \in M$.

(iii) $h(S'', S') = 1$ *for all* $S'' \in M$.

We note that Theorem III sets up, in particular, an isomorphism between the character group of M and the quotient group $\mathfrak{S}/M$, where $\mathfrak{S}$ is the group of all invariance classes. This is doubtless a very special case of the isomorphism set up by Tate (12) between the group of homogeneous spaces of a given abelian variety and the group of characters on the points of the dual variety, where everything must be defined over a given $\mathfrak{p}$-adic field. I conjecture that it is possible to define $h(S', S'')$ in such a way that Theorem III remains true even when the $\mathfrak{d}$ are not defined over $\mathfrak{K}$.

We denote the precise power of $\mathfrak{p}$ dividing ν by $\mathfrak{p}^{\nu}$ and write

$$P = \text{absolute norm } (\mathfrak{p}^{\nu}).$$

We denote the order of the group of the $\mathfrak{d}$ by N.

Lemma 12. *The order of $\mathfrak{G}/\nu\mathfrak{G}$ is NP.*

This follows a well-known pattern (e. g. Cassels (2) or Tate (12)). The kernel of the mapping $\mathfrak{G} \to \nu\mathfrak{G}$ is of order N since, by hypothesis, all the $\mathfrak{d}$ are defined over $\mathfrak{K}$. The theory of the $\mathfrak{p}$-adic $\wp$-function shows that there is a subgroup $\mathfrak{G}_1$ of $\mathfrak{G}$ of finite index which is isomorphic to the $\mathfrak{p}$-adic integers. Hence the mapping $\mathfrak{G}_1 \to \nu\mathfrak{G}_1$ has no kernel and the order of $\mathfrak{G}_1/\nu\mathfrak{G}_1$ is P. This proves the Lemma.

Lemma 13. *The order of the group of all invariance systems is $(NP)^2$.*

For let $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ be a basis of the group of the $\mathfrak{d}$. Let the order of $\mathfrak{e}_j$ be e_j, so

$$\prod_j e_j = N. \tag{1}$$

An invariance class is completely given by the specification for each j of a class of $\mathfrak{K}^*/(\mathfrak{K}^*)^{e_j}$. But now the hypothesis that the $\mathfrak{d}$ are defined over $\mathfrak{K}$ implies that all the e_j-th roots of 1 are in K (cf. (7) of § 2), and so the order of $\mathfrak{K}^*/(\mathfrak{K}^*)^{e_j}$ is $e_j^2 P_j$, where P_j is the absolute norm of the highest power $\mathfrak{p}^{\nu_j}$ of $\mathfrak{p}$ which divides e_j [cf. e. g. Chevalley (3). The proof is similar to that of Lemma 12 but uses the $\mathfrak{p}$-adic logarithm]. The truth of Lemma 13 now follows from (1), since $\prod_j P_j = P^2$ by condition (I) of Theorem I.

Lemma 14. *Let S' be any invariance class other than the trivial one. Then there are invariance classes S'', S''' such that*

$$h(S', S'') \neq 1, \; h(S''', S') = 1.$$

Let $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ be a set of generators of the $\mathfrak{d}$ as in the previous proof. Since the character-group of a finite group is isomorphic to the original group, there is a dual base $\mathfrak{f}_1, \ldots, \mathfrak{f}_s$ of the $\mathfrak{d}$ such that $\mathfrak{f}_j$ has the same order e_j as $\mathfrak{e}_j$ and

$$\chi(\mathfrak{e}_j, \mathfrak{f}_j) = \text{primitive } e_j\text{-th root of } 1$$
$$\chi(\mathfrak{e}_j, \mathfrak{f}_k) = 1 \qquad (j \neq k).$$

Let $\{A'_{\mathfrak{b}}\}$ be an invariant system belonging to S'. Since S' is not trivial, we may suppose, without loss of generality, that

$$A'_{\mathfrak{e}_1} \notin \mathfrak{K}.$$

Hence, by the local class field theory, we may choose an element $a \in \mathfrak{K}^*$ which is not a norm for $\mathfrak{K}(A'_{\mathfrak{e}_1})/\mathfrak{K}$. Then

$$A''_{\mathfrak{f}_1} = a^{1/e_1}$$
$$A''_{\mathfrak{f}_j} = 1 \qquad (2 \leqq j \leqq s)$$

specifies an invariant system $\{A''_{\mathfrak{b}}\}$ in a class S'' (say), say by the rule (15″) (15‴) of § 2.

Now in the representative cocycle for $h(S', S'')$ given by (6) of § 4, the element

$$A''_{\mathfrak{b}} A''_{\mathfrak{e}-\mathfrak{b}}/A''_{\mathfrak{e}}$$

depends only on $\mathfrak{b}$ and $\mathfrak{e}$ modulo $\mathfrak{f}_2, \ldots, \mathfrak{f}_s$. Hence the cocycle is trivial when restricted to the group generated by $\sigma'_{\mathfrak{f}_2}, \ldots, \sigma'_{\mathfrak{f}_s}$, in the notation of (4) of § 4, and so can be regarded as a cocycle of ${}^2H(\mathfrak{K}(A'_{\mathfrak{e}_1})/\mathfrak{K})$, because $\mathfrak{K}(A'_{\mathfrak{e}_1})$ is the fixed field of $\sigma'_{\mathfrak{f}_2}, \ldots, \sigma'_{\mathfrak{f}_s}$. But the resulting cocycle in ${}^2H(\mathfrak{K}(A'_{\mathfrak{e}_1})/\mathfrak{K})$ is precisely the cocycle which defines the e_j-th power

norm residue symbol $(A_{\varepsilon_1}'^{\varepsilon_1}, a)$. Since this is not 1, by construction, we have constructed an S'' with the required properties. The construction of S''' is similar. This completes the proof of the Lemma.

The proof of Theorem III is now almost immediate. By Theorem II, $h(S', S'') = 1$ whenever S', S'' are both in M. Since the group ${}^2H(\overline{\mathfrak{K}}/\mathfrak{K})$ to which the $h(S', S'')$ belong is isomorphic to the rationals modulo 1, $h(S', S'')$ for fixed $S' \in M$ may thus be regarded as a character on the group of the invariance classes S'' modulo M. By Lemma 14, distinct $S' \in M$ give rise to distinct characters; and so, by Lemmas 12 and 13, every character of the group $\mathfrak{S}/M$ of the S'' modulo M must occur in this way. Now, let S''' be an invariance class such that

$$h(S''', S^{\mathrm{IV}}) = 1$$

for all $S^{\mathrm{IV}} \in M$. Then $h(S''', S'')$ depends only on S'' modulo M: and so by what has just been said we can find an $S' \in M$ such that

$$h(S', S'') = h(S''', S'') \qquad (\text{all } S'').$$

By Lemma 14 we must have $S''' = S' \in M$, as required. This proves the sufficiency of one of the two conditions in the enunciation; the other is proved similarly.

Corollary 1. *Suppose that $\mathfrak{K}$ is the real field, but that the other hypotheses of Theorem* III *are satisfied. Then the conclusions of the theorem hold. This case can occur only when $\nu = 2$, and then the order of M is 2 and that of the group $\mathfrak{S}$ of invariance classes S is 4.*

Since ν^2 is a rational integer and $\nu \in \mathfrak{K}$, by hypothesis, ν cannot be a complex multiplication but must be a rational integer. Since the roots $\mathfrak{d}$ of $\nu\mathfrak{d} = \mathfrak{o}$ are defined over the reals, we must have $\nu = 2$, and C is birationally equivalent over the reals to

$$y^2 = 4(x - e_1)(x - e_2)(x - e_3)$$

with real e_1, e_2, e_3. The rest is trivial.

Corollary 2. *Suppose $\mathfrak{K}$ is the field of complex numbers. Then every invariant system is trivial, and the conclusions of the theorem are vacuously fulfilled.*

Finally, we shall need later the following result about an algebraic number-field and its local completions.

Lemma 15. *Suppose that C is defined over an algebraic number-field $\mathfrak{K}$ of finite degree over the rationals. For each valuation $\mathfrak{p}$ of $\mathfrak{K}$ let a suffix $\mathfrak{p}$ denote the corresponding object for the $\mathfrak{p}$-adic closure $\mathfrak{K}_\mathfrak{p}$ of $\mathfrak{K}$. Then, for all except a finite number of $\mathfrak{p}$, the group $M_\mathfrak{p}$ consists of precisely those invariance systems $S_\mathfrak{p}$ for which $\mathfrak{p}$ does not ramify in the associated algebra $\mathfrak{M}(S_\mathfrak{p})$ (Cf. Lang-Tate* (6)).

For it is well-known[31]) that for all except a finite number of $\mathfrak{p}$ the prime $\mathfrak{p}$ does not ramify in $\mathfrak{M}(S_\mathfrak{p})$ for any $S_\mathfrak{p} \in M_\mathfrak{p}$. Since the order of $M_\mathfrak{p}$ and the order of the group of unramifying $S_\mathfrak{p}$ are both N for all but a finite number of $\mathfrak{p}$, the result follows (cf. Lemma 12 and 13).

---

[31]) The proof is roughly as follows. Take those $\mathfrak{p}$ such that (i) ν is a $\mathfrak{p}$-adic unit and (ii) a fixed set of equations for C when taken modulo $\mathfrak{p}$ give a curve $C^{(\mathfrak{p})}$ of genus 1 over the field $\mathfrak{K}^{(\mathfrak{p})}$ of residue s mod $\mathfrak{p}$. Then the map $C^{(\mathfrak{p})} \to C^{(\mathfrak{p})}$ given by $\xi_1 \to \nu\xi_1 = \xi$ has a kernel consisting of N distinct points, defined over the algebraic closure of $\mathfrak{K}^{(\mathfrak{p})}$. This implies by Hensel's lemma that if $\mathfrak{a}_\mathfrak{p}$ is any point on C defined over $\mathfrak{K}_\mathfrak{p}$ and if $\nu\mathfrak{a}_{1,\mathfrak{p}} = \mathfrak{a}_\mathfrak{p}$ then the field $\mathfrak{K}_\mathfrak{p}(\mathfrak{a}_{1,\mathfrak{p}})$ is unramified over $\mathfrak{K}_\mathfrak{p}$.

§ 6. Local and global extensibility.

In this section we first prove the following theorem.

Theorem IV. *Let $\mathfrak{K}$ be an algebraic number-field over the rationals and suppose that conditions* (I) *and* (II) *of Theorem* I *are satisfied. Suppose that a given ν-covering U can be extended to a ν^2-covering $\mathcal{V}_\mathfrak{p}$ defined over $\mathfrak{K}_\mathfrak{p}$ for each local closure $\mathfrak{K}_\mathfrak{p}$ of $\mathfrak{K}$. Then it can be extended to a ν^2-covering defined over $\mathfrak{K}$.*

It is easy to see that condition (I) of Theorem I plays no essential part in the proof and may be dispensed with. I conjecture that condition (II) is also unnecessary, but this I have not been able to prove.

After Lemma 8 it will be enough to prove.

Proposition 4. *Suppose that the conditions of the Theorem* I *are satisfied. Then for each $\mathfrak{d}$ with $\nu\mathfrak{d} = \mathfrak{o}$ there is a divisor $\mathfrak{D}(\mathfrak{d})$ on the curve $\mathcal{D} = \mathcal{D}(U)$ belonging to the ν-covering U which is defined over $\mathfrak{K}$ and which is linearly equivalent to $(\mathfrak{X})^{-1}(\mathfrak{X} + \mathfrak{d})$, where $\mathfrak{X}$ is a generic point on $\mathcal{D}$.*

Here $\mathfrak{X} + \mathfrak{d}$ is intended in the sense of Lemma 5 of § 2. That Lemma then shows that $\mathfrak{D}(\mathfrak{d})$ is linearly equivalent to $(\mathfrak{A})^{-1}(\mathfrak{A} + \mathfrak{d})$, where $\mathfrak{A}$ is any point on $\mathcal{D}$.

Let $\mathfrak{O}$ be any point on $\mathfrak{D}$ defined over $\bar{\mathfrak{K}}$. Then there is a finite normal extension $\mathfrak{L}$ of $\mathfrak{K}$ over which $\mathfrak{O}$ is defined. Let ϱ be any automorphism of $\mathfrak{L}/\mathfrak{K}$. Then

$$\varrho(\mathfrak{O} + \mathfrak{d}) = \varrho\mathfrak{O} + \mathfrak{d},$$

since $\mathfrak{d}$ is defined over $\mathfrak{K}$. Hence for each ϱ there exists a function $\psi_\varrho(\mathfrak{X}) \in \mathfrak{L}(\mathfrak{X})$ having a divisor $\{\varrho(\mathfrak{O} + \mathfrak{d})\}\{\varrho\mathfrak{O}\}^{-1}(\mathfrak{O})(\mathfrak{O} + \mathfrak{d})^{-1}$. We put $\psi_\varrho(\mathfrak{X}) = 1$ if ϱ is the identity automorphism. Then

$$\frac{\psi_\varrho(\mathfrak{X})\,\varrho\{\psi_{\varrho^{-1}\sigma}(\mathfrak{X})\}}{\psi_\sigma(\mathfrak{X})} = c_{\varrho,\sigma} \in \mathfrak{L} \tag{1}$$

is independent of $\mathfrak{X}$, where ϱ, σ are any two automorphisms. By (1), $c_{\varrho,\sigma}$ is a cocycle for the group Γ of automorphisms.

We now show that $c_{\varrho,\sigma}$ is locally a coboundary. Let $\mathfrak{p}$ be any valuation of $\mathfrak{K}$. Then the tensor product $\mathfrak{L}_\mathfrak{p}$ of $\mathfrak{L}$ and $\mathfrak{K}_\mathfrak{p}$ over $\mathfrak{K}$ is a normal abelian algebra over $\mathfrak{K}_\mathfrak{p}$. By hypothesis there exists an extension

$$\mathcal{V}_\mathfrak{p}: \quad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{E}_\mathfrak{p} \arrow[r] & \mathcal{D} \arrow[ur] & \end{tikzcd} \tag{2}$$

where the isomorphisms are defined over the algebraic closure of $\mathfrak{K}_\mathfrak{p}$ but the remaining maps are defined over $\mathfrak{K}_\mathfrak{p}$. Let $\mathbf{X}_\mathfrak{p}$ be a generic point of $\mathcal{E}_\mathfrak{p}$ and $\mathfrak{X}_\mathfrak{p}$ the corresponding point of $\mathcal{D}$. The inverse image on $\mathcal{E}_\mathfrak{p}$ of the divisor $(\mathfrak{O})^{-1}(\mathfrak{O} + \mathfrak{d})$ on $\mathcal{D}$ is defined over $\mathfrak{K}_\mathfrak{p}$ and linearly equivalent to 0. Hence there exists a function $\Lambda_\mathfrak{p}(\mathbf{X}_\mathfrak{p}) \in \mathfrak{K}_\mathfrak{p}(\mathbf{X}_\mathfrak{p})$ with this divisor, and so

$$\frac{\Lambda_\mathfrak{p}(\mathbf{X}_\mathfrak{p})\,\psi_\varrho(\mathfrak{X}_\mathfrak{p})}{\varrho\Lambda_\mathfrak{p}(\mathbf{X}_\mathfrak{p})} = \lambda_{\mathfrak{p},\varrho} \in \mathfrak{L}_\mathfrak{p}, \tag{3}$$

on comparing zeros and poles. On substituting $\mathfrak{X}_\mathfrak{p}$ for $\mathfrak{X}$ in (1) and using (3) we have

$$c_{\varrho,\sigma} = \lambda_{\mathfrak{p},\varrho}\{\varrho\lambda_{\mathfrak{p},\varrho^{-1}\sigma}\}/\lambda_{\mathfrak{p},\sigma},$$

and hence $c_{\varrho,\sigma}$ is a coboundary for $\mathfrak{L}_\mathfrak{p}/\mathfrak{K}$.

By a fundamental theorem of class-field theory (e. g. Chevalley (**3**)), it follows that $c_{\varrho,\sigma}$ is a global coboundary, that is

$$c_{\varrho,\sigma} = \lambda_\varrho \{\varrho \lambda_{\varrho^{-1}\sigma}\}/\lambda_\sigma$$

with the $\lambda_\varrho \in \mathfrak{L}$.

Thus $\lambda_\varrho^{-1} \psi_\varrho(\mathfrak{X})$ as ϱ runs through the relative automorphisms, is a cocycle for $\mathfrak{L}(\mathfrak{X})/\mathfrak{K}(\mathfrak{X})$. Hence, by Noether's generalization of Hilbert's Satz 90 (cf. Chevalley (**3**)), $\lambda_\varrho^{-1} \psi_\varrho(\mathfrak{X})$ is a coboundary, that is

$$\lambda_\varrho^{-1} \psi_\varrho(\mathfrak{X}) = \varrho \Psi(\mathfrak{X})/\Psi(\mathfrak{X}) \tag{4}$$

for some $\Psi(\mathfrak{X}) \in \mathfrak{L}(\mathfrak{X})$.

Let the divisor of $\Psi(\mathfrak{X})$ be $(\mathfrak{O})^{-1}(\mathfrak{O} + \mathfrak{d})\{\mathfrak{D}(\mathfrak{d})\}^{-1}$. Then by (4) and the definition of $\Psi_\varrho(\mathfrak{X})$ it follows that

$$\varrho \mathfrak{D}(\mathfrak{d}) = \mathfrak{D}(\mathfrak{d}).$$

Since this is true for every relative automorphism ϱ, this proves the Proposition and so completes the proof of Theorem IV.

§ 7. The global theory.

In this section $\mathfrak{K}$ is a finite algebraic extension of the rationals, and we make the restrictions (I), (II) of Theorem I. As in § 1, we denote[32]) by $M^{(1)}$ the set of those invariance classes S such that there is a point $\mathfrak{A}_\mathfrak{p}$ on[33]) $\mathcal{D}(S)$ for each local closure $\mathfrak{K}_\mathfrak{p}$ of $\mathfrak{K}$. Since Lemma 6 applies to each $\mathfrak{K}_\mathfrak{p}$, the set $M^{(1)}$ is, in fact, a subgroup of the group of all invariance classes. By Lemma 15, there is only a finite number of primes which can ramify in the algebra $\mathfrak{M}(S)$ for $S \in M^{(1)}$. Hence $M^{(1)}$ contains only a finite number of invariance classes.

We denote, further, by $M^{(2)}$ the set of invariance classes S for which there is an extension $\mathcal{V}$ of $U(S)$

$$\mathcal{V}: \quad \begin{tikzcd} C \arrow[r] \arrow[d, leftrightarrow] & C \arrow[r] \arrow[d, leftrightarrow] & C \\ \mathcal{E} \arrow[r] & \mathcal{D} \arrow[ur] & \end{tikzcd} \tag{1}$$

such that there is a point $\mathbf{A}_\mathfrak{p}$ on $\mathcal{E}$ defined over each $\mathfrak{K}_\mathfrak{p}$. By Lemma 11, Corollary 1, applied to the $\mathfrak{K}_\mathfrak{p}$, $M^{(2)}$ is a subgroup of $M^{(1)}$.

Let $S' \in M^{(1)}$ so $\mathfrak{D}(S')$ contains a point $\mathfrak{A}_\mathfrak{p}$ defined over $\mathfrak{K}_\mathfrak{p}$ for each valuation $\mathfrak{p}$. Hence, by the remarks of § 1, for each $\mathfrak{p}$ there is a ν^2-covering $\mathcal{V}_\mathfrak{p}$, defined over $\mathfrak{K}_\mathfrak{p}$, which extends $U(S')$. Hence, by Theorem IV, there exists a ν^2-covering $\mathcal{V}_0(S')$ defined over $\mathfrak{K}$ which extends $U(S)$. The most general ν^2-covering $\mathcal{V}$, defined over $\mathfrak{K}$, which extends $U(S')$ is then given by Lemma 8, which gives a $1-1$ correspondence[34]) between the invariance classes S and the extensions $\mathcal{V}(S', S)$ (say) of $U(S')$. Further, S' belongs to $M^{(2)}$ if and only if we can determine S so that $\mathcal{E}\{\mathcal{V}(S', S)\}$ contains a point $\mathbf{A}_\mathfrak{p}$ defined over each $\mathfrak{K}_\mathfrak{p}$.

[32]) In § 1, $M^{(1)}$ was defined as a set of ν-coverings. Since the group of ν-coverings is naturally isomorphic to the group of invariance classes by Lemma 4, this is only a change of emphasis.

[33]) Of course, by $U(S)$ we mean the covering correlated to S by Lemma 4, and $\mathcal{D}(S) = \mathcal{D}\{U(S)\}$.

[34]) Not however, as was remarked at the time, a natural $1-1$ correspondence.

We now define a mapping $T(S', S'')$ of pairs $S', S'' \in M^{(1)}$ into the rationals modulo 1. Let $\mathcal{V}_0(S')$ be any global extension of $U(S')$, let $\mathfrak{A}_\mathfrak{p}$ be any point on $\mathcal{D}(S')$ defined over $\mathfrak{K}_\mathfrak{p}$, and let $\Sigma\{\mathcal{V}_0(S'), \mathfrak{A}_\mathfrak{p}\}$ be the invariance class for $\mathfrak{K}_\mathfrak{p}$ defined in Lemma 9, Corollary 2. Consider the cocycle

$$h[S'', \Sigma\{\mathcal{V}_0(S'), \mathfrak{A}_\mathfrak{p}\}] \in {}^2H(\bar{\mathfrak{K}}_\mathfrak{p}/\mathfrak{K}_\mathfrak{p}) \tag{2}$$

given by Theorem II, with $\mathfrak{K}_\mathfrak{p}$ for $\mathfrak{K}$.

Lemma 16. *For given S', S'' and $\mathcal{V}_0(S')$ there is only a finite number of $\mathfrak{p}$ for which* (2) *is non-trivial.*

All but a finite number of $\mathfrak{p}$ have the three properties:

(i) ν is a $\mathfrak{p}$-adic unit.

(ii) $\mathfrak{p}$ does not ramify in a field of definition for all the maps in (1), with $\mathcal{D} = \mathcal{D}(S')$, $\mathcal{E} = \mathcal{E}(\mathcal{V}_0(S'))$.

(iii) There is a set of equations for C which, when reduced modulo $\mathfrak{p}$ define a curve $C^{(\mathfrak{p})}$ of genus 1 over the field $\mathfrak{K}^{(\mathfrak{p})}$ of residues modulo 1.

(iv) $\mathfrak{p}$ is not exceptional in Lemma 15.

We shall show that (2) is trivial if (i), (ii), (iii) and (iv) hold. Then, by a result of Lang and Tate (6) already quoted, if $\mathfrak{a}_\mathfrak{p}$ is any point of C defined over $\mathfrak{K}_\mathfrak{p}$ the points $\mathfrak{a}_{2,\mathfrak{p}}$ with $\nu^2\mathfrak{a}_{2,\mathfrak{p}} = \mathfrak{a}_\mathfrak{p}$ are all defined over an unramified extension of $\mathfrak{K}_\mathfrak{p}$. In particular, if (ii) holds as well and $\mathfrak{A}_\mathfrak{p}$ is the point occuring in (2), every point and map of the diagram

$$\begin{tikzcd}
\mathfrak{a}_{2,\mathfrak{p}} \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{a}_{1,\mathfrak{p}} = \nu\mathfrak{a}_{2,\mathfrak{p}} \arrow[r] \arrow[d, leftrightarrow] & \mathfrak{a}_\mathfrak{p} = \nu\mathfrak{a}_{1,\mathfrak{p}} \\
\mathbf{A}_\mathfrak{p} \arrow[r] & \mathfrak{A}_\mathfrak{p} \arrow[ur] &
\end{tikzcd}$$

is defined over an unramified extension of $\mathfrak{K}_\mathfrak{p}$. Hence if $\mathcal{E}(\mathcal{V}_0)$ is given by the equations (15) of § 3, the $\alpha_\mathfrak{b}(\mathfrak{a}_{2,\mathfrak{p}})$ all lie in an unramified extension of $\mathfrak{K}_\mathfrak{p}$. Hence

$$\Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p}) \in M_\mathfrak{p}$$

by the definition of $\Sigma(\mathcal{V}_0, \mathcal{V}_\mathfrak{p})$ (Lemma 9, Corollary 2) and Lemma 15. Further[35])

$$S'' \in M^{(1)} < M_\mathfrak{p}. \tag{3}$$

Hence the left-hand side of (2) is trivial by Theorem II (iii) (or by Theorem III), if (i), (ii), (iii) and (iv) hold; which proves the Lemma.

There is a natural map $i_\mathfrak{p}$ of ${}^2H(\bar{\mathfrak{K}}_\mathfrak{p}/\mathfrak{K}_\mathfrak{p})$ into the rationals mod 1. After Lemma 16 we may write[36])

$$T(S', S'') = \sum_\mathfrak{p} i_\mathfrak{p} h\{S'', \Sigma(\mathcal{V}_0(S'), \mathfrak{A}_\mathfrak{p})\}, \tag{4}$$

since only a finite number of terms on the right differ from 0.

---

[35]) We consider $\mathfrak{K}$ as canonically embedded in each $\mathfrak{K}_\mathfrak{p}$, so $M^{(1)} = \bigcap_\mathfrak{p} M_\mathfrak{p}$.

[36]) It is more convenient here to take the arguments as invariance classes; while in the enunciation of Theorem I they are the ν-coverings to which they canonically correspond.

Lemma 17. *The right hand side of* (4) *depends only on* S', S'' *and not on* $\mathcal{V}_0(S')$ *or the* $\mathfrak{A}_\mathfrak{p}$.

If $\mathcal{V}_0$ is replaced by another extension $\mathcal{V}_1$ of $U(S')$ we have

$$\Sigma(\mathcal{V}_1, \mathfrak{A}_\mathfrak{p}) = S\,\Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})$$

for all $\mathfrak{p}$ by Lemma 9, Corollary 3, where S is a global invariance class. Now

$$\sum_\mathfrak{p} i_\mathfrak{p} h(S'', S) = 0,$$

since this is the sum of the invariants of a global cocycle, (cf. Chevalley (3)). Hence the right hand side of (4) is independent of $\mathcal{V}_0$.

Let $\mathfrak{A}'_\mathfrak{p}$ be any point on $\mathcal{D}(S')$ defined over $\mathfrak{K}_\mathfrak{p}$, so $\mathfrak{A}'_\mathfrak{p} = \mathfrak{A}_\mathfrak{p} + \mathfrak{b}_\mathfrak{p}$, where $\mathfrak{b}_\mathfrak{p}$ is on C. By Lemma 10 we have

$$\Sigma(\mathcal{V}_0, \mathfrak{A}'_\mathfrak{p}) = \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p} + \mathfrak{b}_\mathfrak{p}) = \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\,\Sigma(\mathfrak{b}_\mathfrak{p}),$$

where $\Sigma(\mathfrak{b}_\mathfrak{p})$ is the invariance class corresponding to $\mathfrak{b}_\mathfrak{p}$ according to Lemma 3, so $\Sigma(\mathfrak{b}_\mathfrak{p}) \in M_\mathfrak{p}$ by the definition of $M_\mathfrak{p}$. Since $S'' \in M^{(1)} < M_\mathfrak{p}$, the element

$$h(S'', \Sigma(\mathfrak{b}_\mathfrak{p}))$$

of ${}^2H(\mathfrak{K}'_\mathfrak{p}/\mathfrak{K}_\mathfrak{p})$ is trivial, by Theorem II (iii) (or by Theorem III). Hence the right hand side of (4) is independent of the choice of the $\mathfrak{A}_\mathfrak{p}$. This concludes the proof of the Lemma.

Lemma 18.

$$T(S', S''S''') = T(S', S'') + T(S', S'''), \tag{5}$$

$$T(S'S''', S'') = T(S', S'') + T(S''', S'') \tag{6}$$

for any $S', S'', S''' \in M^{(1)}$.

The equation (5) follows at once from the definition (4) and Theorem II (i), so it remains only to prove (6). Let $\mathcal{V}'_0$, $\mathcal{V}''_0$ be extensions of $U(S')$, $U(S''')$ respectively and let $\mathcal{V}_0$ be the extension of $\mathcal{V}(S'S''')$ given by Lemma 11. For each $\mathfrak{p}$ let $\mathfrak{A}'_\mathfrak{p}$, $\mathfrak{A}'''_\mathfrak{p}$ be points on $\mathcal{D}(S')$, $\mathcal{D}(S''')$ respectively, defined over $\mathfrak{K}_\mathfrak{p}$ and let $\mathfrak{A}_\mathfrak{p} = \mathfrak{A}'_\mathfrak{p} + \mathfrak{A}''_\mathfrak{p}$ be the corresponding points on $\mathcal{D}(S'S'')$ given by Lemma 6, and the note after its enunciation. Then by Lemma 11, Corollary 2 we have

$$\Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p}) = \Sigma(\mathcal{V}'_0, \mathfrak{A}'_\mathfrak{p})\,\Sigma(\mathcal{V}'''_0, \mathfrak{A}'''_\mathfrak{p}),$$

so

$$h\{S'', \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\} = h\{S'', \Sigma(\mathcal{V}'_0, \mathfrak{A}'_\mathfrak{p})\}\, h\{S'', \Sigma(\mathcal{V}'''_0, \mathfrak{A}'''_\mathfrak{p})\}. \tag{7}$$

Now (6) follows at once from (4) and (7).

Lemma 19.

$$T(S, S) = 0 \tag{8}$$

for all $S \in M^{(1)}$.

By Theorem II, Corollary 1 there is an element of $h(S, \mathcal{V}_0)$ of the global group ${}^2H(\bar{\mathfrak{K}}/\mathfrak{K})$ with the property that $h\{S, \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\}$ is the natural image[37]) of $h(S, \mathcal{V}_0)$ in ${}^2H(\bar{\mathfrak{K}}_\mathfrak{p}/\mathfrak{K}_\mathfrak{p})$ for each point $\mathfrak{A}_\mathfrak{p}$ on $\mathcal{D}(S)$ defined over $\mathfrak{K}_\mathfrak{p}$. Hence we have

[37]) We identify $\bar{\mathfrak{K}}$ with a subfield of $\bar{\mathfrak{K}}_\mathfrak{p}$.

$$T(S, S) = \sum_{\mathfrak{p}} i_{\mathfrak{p}} h\{S, \Sigma(\mathcal{V}_0, \mathfrak{A}_{\mathfrak{p}})\}$$

$$(9) \qquad = \sum_{\mathfrak{p}} i_{\mathfrak{p}} h(S, \mathcal{V}_0)$$

$$= 0,$$

because it is the sum of the invariants of a global cocycle.

Corollary. *For any $S', S'' \in M^{(1)}$ we have*

$$(10) \qquad T(S', S'') + T(S'', S') = 0.$$

This follows at once from the two preceding lemmas.

Before completing the proof of Theorem I we require the following rather deep lemma. The proof follows closely the proof of a special case in the first paper of this series.

Lemma 20. *For each $\mathfrak{p}$ let an invariance class $S_{\mathfrak{p}}$ for $\mathfrak{p}$ be given, and suppose that $\mathfrak{p}$ ramifies in the algebra $\mathfrak{M}(S_{\mathfrak{p}})$ corresponding to $S_{\mathfrak{p}}$ for at most a finite number of $\mathfrak{p}$. A necessary and sufficient condition that there exist an invariance class S for $\mathfrak{K}$ such that*[38])

$$(11) \qquad S S_{\mathfrak{p}}^{-1} \in M_{\mathfrak{p}}$$

for every $\mathfrak{p}$ is that

$$(12) \qquad \sum_{\mathfrak{p}} i_{\mathfrak{p}} h(S'', S_{\mathfrak{p}}) = 0$$

for every $S'' \in M^{(1)}$.

In the first place, if $S'' \in M^{(1)} < M_{\mathfrak{p}}$ and (11) holds, we have

$$h(S'', S S_{\mathfrak{p}}^{-1}) = 1$$

by (II) and Theorem III, that is

$$i_{\mathfrak{p}} h(S'', S) = i_{\mathfrak{p}} h(S'', S_{\mathfrak{p}}).$$

But

$$(13) \qquad \sum_{\mathfrak{p}} i_{\mathfrak{p}} h(S'', S) = 0$$

because $h(S'', S)$ is a globally defined cocycle, so (12) holds.

Now suppose, conversely, that (12) holds. Let Π be a finite set of valuation with the properties:

(i) Π contains all archimedean $\mathfrak{p}$.

(ii) Π contains all $\mathfrak{p}$ for which ν is not a $\mathfrak{p}$-adic unit.

(iii) Π contains all the exceptional $\mathfrak{p}$ of Lemma 15.

(iv) The algebras $\mathfrak{M}(S_{\mathfrak{p}})$ associated with the given invariante classes $S_{\mathfrak{p}}$ are unramified at all $\mathfrak{p} \notin \Pi$.

(v) Let $\mathfrak{e}$ be a divisor of the exponent n of the group of $\mathfrak{d}$ with $\nu\mathfrak{d} = \mathfrak{o}$. If $u \in \mathfrak{K}$ has the property that

$$\left.\begin{array}{ll} u \in (\mathfrak{K}_{\mathfrak{p}})^{\mathfrak{e}} & (\text{all } \mathfrak{p} \in \Pi) \\ u \text{ is a } p\text{-adic unit} & (\text{all } \mathfrak{p} \notin \Pi) \end{array}\right\},$$

then $u \in (\mathfrak{K})^{\mathfrak{e}}$.

---

[38]) We recall that $M_{\mathfrak{p}}$ is the group of invariance classes $S''_{\mathfrak{p}}$ for $\mathfrak{K}_{\mathfrak{p}}$ such that there is a point on $D(S''_{\mathfrak{p}})$ defined over $\mathfrak{K}_{\mathfrak{p}}$.

Such sets Π exist (cf. Chevalley (3)). Let $\mathbf{S}_\mathfrak{p}$ denote the group of invariant classes for $\mathfrak{K}_\mathfrak{p}$ and denote by $\mathbf{I}$ the direct product of the $\mathbf{S}_\mathfrak{p}(\mathfrak{p} \in \Pi)$. We intend to prove now

$$\text{(14)} \qquad \text{order } \mathbf{I} = N^{2p},$$

where N is the order of the group of the $\mathfrak{d}$ and where p is the number of $\mathfrak{p}$ in Π. Indeed, if $\mathfrak{e}_1, \ldots, \mathfrak{e}_s$ is a basis for the group of the $\mathfrak{d}$, with respective orders $e_1, \ldots, e_s$, the proof of Lemma 13 showed that

$$\text{order of } \mathbf{S}_\mathfrak{p} = \prod_{1 \leq j \leq s} \{\text{order of } \mathfrak{K}_\mathfrak{p}^*/(\mathfrak{K}_\mathfrak{p}^*)^{e_j}\}.$$

On the other hand it is well known[39] (cf. Chevalley (3)), that if Π satisfies the conditions (i)—(v), then

$$\prod_{\mathfrak{p} \in \Pi} \{\text{order of } \mathfrak{K}_\mathfrak{p}^*/(\mathfrak{K}_\mathfrak{p}^*)^{e_j}\} = e_j^{2p}.$$

Since $\prod_j e_j = N$, the result (14) follows.

Now let $\mathbf{N}$ consist of those elements of $\mathbf{I}$ whose $\mathfrak{p}$-th coordinate is in $M_\mathfrak{p}$ for all $\mathfrak{p} \in \Pi$. By Lemmas 12 and 13 and by (14) we have

$$\text{(15)} \qquad \text{order } \mathbf{N} = N^p.$$

Let $\mathbf{L}$ denote the group of all invariance classes for $\mathfrak{K}$ which are unramified for every $\mathfrak{p} \notin \Pi$. If $\mathfrak{K}_\Pi$ denotes the multiplicative group of elements of $\mathfrak{K}$ which are units for every $\mathfrak{p} \notin \Pi$, we have

$$\text{order } \mathbf{L} = \prod_{1 \leq j \leq s} (\text{order } \mathfrak{K}_\Pi/(\mathfrak{K}_\Pi)^{e_j})$$

(cf. (15″), (15‴) of § 2). It is well-known (cf. Chevalley (3)) that under the conditions (i)—(v) we have

$$\text{order } \mathfrak{K}_\Pi/(\mathfrak{K}_\Pi)^{e_j} = (e_j)^p,$$

so

$$\text{(16)} \qquad \text{order } \mathbf{L} = N^p.$$

There is a natural mapping of $\mathbf{L}$ into each of the groups $\mathbf{S}_\mathfrak{p}$ of invariance classes for $\mathfrak{K}_\mathfrak{p}$ and so into $\mathbf{I}$. Let $\mathbf{Q}$ be the image of $\mathbf{L}$ in this mapping. By (v) in the definition of Π we have[40]

$$\text{(17)} \qquad \mathbf{Q} \cong \mathbf{L}.$$

Clearly

$$\text{(18)} \qquad \mathbf{Q} \cap \mathbf{N} \cong M^{(1)}.$$

To every element $S'' \in M^{(1)}$ there corresponds an additive character $\omega = \omega_{S''}$ on $\mathbf{I}$ defined by

$$\text{(19)} \qquad \omega_{S''}(J) = \sum_{\mathfrak{p} \in \Pi} i_\mathfrak{p} h(S'', S_\mathfrak{p}),$$

where J is an element of $\mathbf{I}$ with components $S_\mathfrak{p}$. If S'' is such that $\omega_{S''}(J) = 0$ for every $J \in \mathbf{I}$, the invariance class S'' is trivial in every $\mathfrak{K}_\mathfrak{p}(\mathfrak{p} \in \Pi)$, by Lemma 14; and so S'' is trivial in $\mathfrak{K}$ by condition (v) in the definition of Π. Hence

$$\text{(20)} \qquad \mathbf{\Omega} \cong M^{(1)},$$

where $\mathbf{\Omega}$ is the group of characters $\omega_{S''}$ with $S'' \in M^{(1)}$.

---

[39]) The reader is reminded that Condition II of Theorem I implies that all the e_j-th roots of 1 are in $\mathfrak{K}$.

[40]) $\cong$ means "is isomorphic to".

By Theorem III, if $J \in \mathbf{N}$ with components $S_\mathfrak{p}$, we have

$$\omega_{S''}(J) = \sum_{\mathfrak{p} \in \Pi} i_\mathfrak{p} h(S'', S_\mathfrak{p}) = \sum_{\mathfrak{p} \in \Pi} 0 = 0, \tag{21}$$

since $S_\mathfrak{p} \in M_\mathfrak{p}$, $S'' \in M^{(1)} < M_\mathfrak{p}$ by the definitions of $\mathbf{N}$, $M^{(1)}$ respectively. Similarly, if $J \in \mathbf{Q}$, so J corresponds to an invariant class S for $\mathfrak{K}$ which does not ramify outside Π, we have

$$\omega_{S''}(J) = \sum_{\mathfrak{p} \in \Pi} i_\mathfrak{p} h(S'', S) = \sum_{\text{all } \mathfrak{p}} i_\mathfrak{p} h(S'', S) = 0, \tag{22}$$

since $h(S'', S)$ is a globally defined cocycle. Hence

$$\omega_{S''}(J) = 0 \text{ (all } J \in \mathbf{QN}, \text{ all } \omega_{S''} \in \mathbf{\Omega}) \tag{23}$$

where $\mathbf{QN}$ is the subgroup of $\mathbf{I}$ generated by the elements of $\mathbf{Q}$ and $\mathbf{N}$. By (15), (16), (17), (18) we have

$$\text{order } \mathbf{QN} = \frac{N^{2p}}{\text{order } M^{(1)}}. \tag{24}$$

By (14) and (20) we have

$$\frac{\text{order } \mathbf{I}}{\text{order } \mathbf{\Omega}} = \frac{N^{2p}}{\text{order } M^{(1)}}. \tag{25}$$

The coincidence of the orders in (24) and (25) together with (23) shows that $\mathbf{QN}$ is precisely the set of $J \in \mathbf{I}$ for which $\omega(J) = 0$ (all $\omega \in \mathbf{\Omega}$). On recollecting the definitions of $\mathbf{\Omega}$, $\mathbf{Q}$, $\mathbf{N}$, one sees that this is just the assertion of Lemma 20.

The following Lemma will conclude the proof of Theorem I.

Lemma 21. *A necessary and sufficient condition that $S \in M^{(2)}$ is that*

$$T(S'', S) = 0 \tag{26}$$

for all $S'' \in M^{(1)}$.

Suppose, first, that $S \in M^{(2)}$. By the definition of $M^{(2)}$ we may choose $\mathcal{V}_0$ in (4) so that there is a point $\mathbf{A}_\mathfrak{p}$ on $\mathcal{E}(\mathcal{V}_0)$ defined over $\mathfrak{K}_\mathfrak{p}$ for each $\mathfrak{p}$. We now choose $\mathfrak{A}_\mathfrak{p}$ to be the image of $\mathbf{A}_\mathfrak{p}$ under the map $\mathbf{X} \to \mathfrak{X}$ of $\mathcal{E}(\mathcal{V}_0) \to \mathcal{D}(S)$. Then $\Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})$ is the trivial invariance class for each $\mathfrak{p}$, by the last sentence of Lemma 9, Corollary 2. Hence $h\{S'', \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\}$ is trivial for each $\mathfrak{p}$, and so $T(S'', S) = 0$, by (4).

We now suppose that (26) holds and wish to show that $S \in M^{(2)}$. In Lemma 20 put $S_\mathfrak{p} = \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})$. Then, by (4), the condition (26) is just the same as the condition (12). Hence, if (26) holds for all $S'' \in M^{(1)}$ there exists an invariance class S' for $\mathfrak{K}$ such that

$$S' \{\Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\}^{-1} \in M_\mathfrak{p}$$

for all $\mathfrak{p}$; that is, in the notation of Lemma 10

$$S' = \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p})\, \Sigma(\mathfrak{b}_\mathfrak{p})$$

for some $\mathfrak{b}_\mathfrak{p}$ on C defined over $\mathfrak{K}_\mathfrak{p}$. On replacing $\mathfrak{A}_\mathfrak{p}$ by $\mathfrak{A}_\mathfrak{p} + \mathfrak{b}_\mathfrak{p}$ we may suppose by Lemma 10 that

$$S' = \Sigma(\mathcal{V}_0, \mathfrak{A}_\mathfrak{p}) \qquad \text{(all } \mathfrak{p}). \tag{27}$$

Let $\mathcal{V}_1$ be the covering which corresponds to S'^{-1} in the sense of Lemma 8, that is $\{\mathfrak{A}_\mathfrak{b}\} \in S'^{-1}$ in (12) of § 3 when the $\alpha_\mathfrak{b}(\mathfrak{x}_2)$ corresponding to the extension $\mathcal{V}_0$. Then, by Lemma 9, Corollary 3 and (27) we have

$$\Sigma(\mathcal{V}_1, \mathfrak{A}_\mathfrak{p}) = \text{trivial} \qquad (\text{all } \mathfrak{p}).$$

By the last sentence of Lemma 9, Corollary 2 it follows that each $\mathfrak{A}_\mathfrak{p}$ is the image under $\mathcal{E}(\mathcal{V}_1) \to \mathcal{D}(S')$ of a point $\mathbf{A}_\mathfrak{p}$ of $\mathcal{E}(\mathcal{V}_1)$ defined over $\mathfrak{K}_\mathfrak{p}$. In particular, the points $\mathbf{A}_\mathfrak{p}$ exist, so $S \in M^{(2)}$, by the definition of $M^{(2)}$.

Finally, we note that the assertions of Theorem I are all contained in Lemma 16—19 and 21.

Appendix.

I prove here a simple result in group cohomology for which I know no reference.

Lemma. *Let A be a Γ-module (written additively) where Γ is a finite group. Let Δ be a subgroup of Γ and let $\tau_1 =$ identity, $\tau_2, \ldots, \tau_t$ be a set of coset representatives for Γ modulo Δ:*

$$\Gamma = \sum_i \tau_i \Delta. \tag{1}$$

Suppose that

$$A = \sum_i \tau_i B \qquad (\textit{direct}), \tag{2}$$

where B is a Δ-sub-module of A. Then there is a natural isomorphism between the groups ${}^jH(\Delta, B)$ and ${}^jH(\Gamma, A)$ for all j.

Suppose, first that $j = 0$. Then ${}^0H(\Gamma, A)$ is the quotient group of the elements of A invariant under Γ by the norms $\left(\sum_{\gamma \in \Gamma} \gamma\right) a$, where $a \in A$. Any Γ-invariant element a of A is of the shape

$$a = \sum_i \tau_i b,$$

by (2), where $b \in B$ is Δ-invariant; and the correspondence $a \longleftrightarrow b$ induces the required isomorphism.

For general $j \neq 0$ we have a canonical isomorphism[41])

$${}^jH(\Gamma, A) \cong {}^0H(\Gamma, K_j[\Gamma] \otimes A),$$

where $K_j[\Gamma]$ is a Γ-module determined by j and Γ. Then

$$K_j[\Gamma] \otimes A = \sum_i \tau_i \{K_j[\Gamma] \otimes B\}$$

and so by the case $j = 0$ we have a natural isomorphism

$$\begin{aligned} {}^jH(\Gamma, A) &\cong {}^0H(\Gamma, K_j[\Gamma] \otimes A) \\ &\cong {}^0H(\Delta, K_j[\Gamma] \otimes B). \end{aligned}$$

But now there are the following natural exact sequences of Δ-modules

$$0 \to W_j \to K_j[\Gamma] \to K_j[\Delta] \to 0 \qquad (j > 0),$$
$$0 \to K_j[\Delta] \to K_j[\Gamma] \to W_j \to 0 \qquad (j < 0),$$

[41]) Cf. Chevalley (3). In Chevalley's notation $K_j[\Gamma]$ is the j-fold tensor product of $J = J[\Gamma]$ with itself if $j > 0$, and the $|j|$-fold tensor product of $I = I[\Gamma]$ with itself if $j < 0$.

where in both cases W_j is a splitting Δ-module[42]). Hence we have the following natural exact sequences of Δ-modules in which U_j is a splitting module:

$$0 \to U_j \to K_j[\Gamma] \otimes B \to K_j[\Delta] \otimes B \to 0,$$
$$0 \to K_j[\Delta] \otimes B \to K_j[\Gamma] \otimes B \to U_j \to 0.$$

In both cases the corresponding exact sequence of homology groups shows that

$${}^0H(\Delta, K_j[\Gamma] \otimes B) \cong {}^0H(\Delta, K_j[\Delta] \otimes B) \cong {}^1H(\Delta, B)$$

(natural isomorphisms). This concludes the proof of the Lemma.

[42]) In the notation of Chevalley Chapter 7, when $j > 0$ the map $K_j[\Gamma] \to K_j[\Delta]$ is induced by the map of $J[\Gamma] \to J[\Delta]$ which maps $\bar{s} \to \bar{s}$ $(s \in \Delta)$ and $\bar{s} \to 0 (s \in \Gamma, s \notin \Delta)$. Chevalley uses latin letters, as s, for the elements of the group. When $j < 0$ the map is that induced by the natural inclusion map of the group ring $\mathbf{Z}[\Delta]$ in $\mathbf{Z}[\Gamma]$.

References.

(1) *E. Artin*, Algebraic numbers and algebraic functions I. Mimeographed lecture notes, Princeton and New York Universities 1950/51.

(2) *J. W. S. Cassels*, Arithmetic on curves of Genus 1. I. On a conjecture of Selmer. J. reine u. angew. Math. **202** (1959), 52—99.

(3) *C. Chevalley*, Class field theory, Nagoya, 1953/4.

(4) *H. Hasse*, Invariante Kennzeichnung relativ-abelscher Zahlkörper mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers. Abh. d. Deutschen Akad. d. Wiss. zu Berlin 1947, 8, 56.

(5) *H. Hasse*, Existenz und Mannigfaltigkeit abelscher Algebren mit vorgegebener Galoisgruppe über einem Teilkörper des Grundkörpers I, II, III. Math. Nachr. **1** (1948), 40—61, 213—217, 277—283.

(6) *S. Lang* and *J. Tate*, Principal homogeneous spaces over abelian varieties. Amer. J. Maths. **80** (1958), 659—684.

(7) *P. Roquette*, Über das Hassesche Klassenkörper-Zerlegungsgesetz. J. reine u. angew. Math. **197** (1957), 49—67.

(8) *E. S. Selmer*, The diophantine equation $ax^3 + by^3 + cz^3 = 0$. Acta Math. (Stockh.) **85** (1951), 203—362.

(9) *E. S. Selmer*, The diophantine equation $ax^3 + by^3 + cz^3 = 0$. Completion of the tables. Acta Math. (Stockh.) **92** (1954), 191—197.

(10) *E. S. Selmer*, The diophantine equation $\eta^2 = \xi^3 - D$. A note on Cassels' Method. Math. Scand. **3** (1955), 68—74.

(11) *J.-P. Serre*, Groupes algébriques et corps de classes. Actualités Sci. Ind. 1264 (1959) (Pub. de l'Inst. de Math. de l'Univ. de Nancago **7**).

(12) *J. Tate*, *WC*-groups over $\mathfrak{p}$-adic fields. Séminaire Bourbaki 1957/1958 Exposé **156**.

(13) *A. Weil*, L'arithmétique sur les courbes algébriques. Acta Math. (Stockh.) **52** (1928), 281—315.

(14) *A. Weil*, On algebraic groups and homogeneous spaces. Amer. J. Math. **77** (1955), 493—512.

(15) *H. Reichardt*, Über die Diophantische Gleichung $ax^4 + bx^2y^2 + cy^4 = ez^2$, Math. Annalen **117** (1940), 235—276

Eingegangen 23. Oktober 1959.

Decompositions of the Plane into Three Sets.

By *Frederick Bagemihl* at South Bend (Ind.), USA.

We shall be concerned with the Euclidean plane $\mathcal{P}$ provided with a rectangular Cartesian coordinate system having a horizontal x-axis and a vertical y-axis. The straight lines that are parallel to the x- (or y-)axis or coincide with it will be called x-(or y-)lines, and the lines that have slope 1 will be called t-lines. If the union of a set of x-(y-, t-)lines intersects the y-axis (x-axis, line $x + y = 0$) in a linear set having a certain property, we say that the set of lines possesses this property; for example, a set of first category of t-lines is a set of t-lines whose union intersects the line $x + y = 0$ in a linear set of first category.

The set of real numbers will be denoted by $\mathcal{E}$. If $A < \mathcal{E}$ and $r \in \mathcal{E}$, we set

$$\{a + r : a \in A\} = A[r].$$

§ 1. Decompositions obtained without the continuum hypothesis.

We are going to make use of the following theorems:

(B_K) *Let $S < \mathcal{E}$, $T < \mathcal{E}$, S be at most enumerable and T be of first category. Then $\mathcal{E}$ contains a residual subset R such that $S[r] \cap T$ is empty for every $r \in R$.*

(B_M) *Let $S < \mathcal{E}$, $T < \mathcal{E}$, S be at most enumerable and T be of measure zero. Then $\mathcal{E}$ contains a subset R such that $\mathcal{E} - R$ is of measure zero and $S[r] \cap T$ is empty for every $r \in R$.*

(B_P) *Let $S < \mathcal{E}$, $T < \mathcal{E}$, S be at most enumerable and T be of power less than $2^{\aleph_0}$. Then $\mathcal{E}$ contains a subset R such that $\mathcal{E} - R$ is of power less than $2^{\aleph_0}$ and $S[r] \cap T$ is empty for every $r \in R$.*

We have proved (B_K) and (B_M) in [1]; (B_P) is proved, by a similar argument that involves power instead of category or measure, with the aid of the fact (see, e. g., [5], p. 7) that the sum of enumerably many cardinal numbers, each of which is less than $2^{\aleph_0}$, is less than $2^{\aleph_0}$. We shall employ actually only a part of the conclusion of these theorems, namely, the existence of a single r with the asserted property.

Theorem 1. *Let n be a natural number, and suppose that $\mathcal{P} = E_1 \cup E_2 \cup E_3$, where every x-line, with at most a set of first category of exceptions, intersects E_1 in at most n points, and every t-line, with at most a set of first category of exceptions, intersects E_2 in a linear nowhere dense set of points. Then there exists a set of y-lines that is everywhere of second category, every member of which intersects E_3 in a linear set of second category.*

Proof. Assume that the conclusion is false. Then there exists an interval J of $\mathcal{E}$, and a subset T of J of first category, such that, for every $j \in J - T$, the line $x = j$ intersects E_3 in a linear set of first category. Let S denote the set of rational numbers. According to Theorem (B_K), there exists a real number h such that $S[h] \cap T$ is empty. If U is the set of those elements of $S[h]$ that belong to the interval J, then U is an

enumerable subset of J with the property that the union of the vertical lines $x = u (u \in U)$ intersects E_3 in a set of points whose (orthogonal) projection on the y-axis is a linear set M of first category. Let the ordinates of the points of M constitute the set C of real numbers. Define Q to be the set of real numbers q with the property that the t-line containing the point on the y-axis whose ordinate is q does not intersect E_2 in a linear nowhere dense set of points. The hypothesis concerning E_2 implies that Q is a subset of $\mathcal{S}$ of first category. Set

$$D = \bigcup_{u \in U} Q[u];$$

then D (which is empty if Q is empty) is the union of enumerably many subsets of $\mathcal{S}$ of first category, and is therefore a subset of $\mathcal{S}$ of first category. Let F be the set of real numbers f such that the line $y = f$ intersects E_1 in more than n points. The hypothesis concerning E_1 implies that F is a subset of $\mathcal{S}$ of first category. Consequently, the set

$$T' = C \cup D \cup F$$

is also a subset of $\mathcal{S}$ of first category. By Theorem (B_K), there exists a real number h' such that $S[h'] \cap T'$ is empty. We write $V = S[h']$.

Define Z to be the set of all points (u, v) of $\mathcal{P}$ with both $u \in U$ and $v \in V$. Since $V \cap C$ is empty, we have

$$Z < E_1 \cup E_2.$$

Suppose that (u_0, v_0) is a point belonging to Z. The t-line that contains this point intersects the y-axis in the point whose ordinate is $v_0 - u_0$. Now $v_0 - u_0$ is not in Q; because if $v_0 - u_0 = q_0 \in Q$, then $v_0 = q_0 + u_0 \in D$, which contradicts the fact that $V \cap D$ is empty. It follows, from the definition of Q, that every t-line containing a point of Z intersects E_2 in a linear nowhere dense set of points.

Any t-line that contains a point of Z has an equation

$$y - v_0 = x - u_0$$

with appropriate $u_0 \in U$ and $v_0 \in V$. This t-line intersects the vertical line $x = u_1$, where $u_1 \in U$, in the point whose ordinate is

$$u_1 + v_0 - u_0,$$

and it intersects the horizontal line $y = v_1$, where $v_1 \in V$, in the point whose abscissa is

$$v_1 + u_0 - v_0.$$

Since $U < S[h]$ and $V = S[h']$, there exist numbers s_0, s_0', s_1, s_1' in S such that

$$u_1 = s_1 + h,\ v_0 = s_0' + h',\ u_0 = s_0 + h,\ v_1 = s_1' + h',$$

and hence

$$u_1 + v_0 - u_0 = (s_1 + s_0' - s_0) + h' \in S[h'],$$

$$v_1 + u_0 - v_0 = (s_1' + s_0 - s_0') + h \in S[h].$$

Thus, every t-line containing a point of Z intersects every vertical line $x = u$, with $u \in U$, in a point of Z, and intersects every horizontal line $y = v$, with $v \in V$, in a point whose abscissa belongs to $S[h]$.

Now select an element v_0^* of V, and $n + 2$ elements $u_0^*, u_1^*, \ldots, u_n^*, u_{n+1}^*$ of U such that

$$u_0^* < u_1^* < \cdots < u_n^* < u_{n+1}^*.$$

For $m = 0, 1, \ldots, n$, the t-line passing through the point (u_m^*, v_0^*) of Z contains the segment L_m whose endpoints are

$$(u_m^*, v_0^*),\ (u_m^* + u_{n+1}^* - u_n^*, v_0^* + u_{n+1}^* - u_n^*).$$

Since $L_m \cap E_2$ is a nowhere dense subset of L_m, the projection of $\bigcup_{m=0}^{n} (L_m \cap E_2)$ on the y-axis is a nowhere dense subset of this axis. Hence, as V is an everywhere dense subset of $\mathcal{S}$, there is an element v^* of V that lies in the open interval of $\mathcal{S}$ whose endpoints are

$$v_0^*, v_0^* + u_{n+1}^* - u_n^*,$$

with the property that the line $y = v^*$ intersects every $L_m (m = 0, 1, \ldots, n)$ in a point not belonging to E_2. The line $y = v^*$ intersects L_m in a points of Z; because the abscissa of this point of intersection is

$$v^* - v_0^* + u_m^*,$$

which as we already know belongs to $S[h]$, and since

$$v^* - v_0^* < u_{n+1}^* - u_n^*,$$

we have

$$u_m^* < v^* - v_0^* + u_m^* < u_m^* + u_{n+1}^* - u_n^* \leqq u_{n+1}^*,$$

so that the abscissa in question is indeed an element of U. This implies, however, due to the relation $Z < E_1 \cup E_2$, that the x-line $y = v^*$ intersects E_1 in at least $n + 1$ points, which contradicts the fact that $V \cap F$ is empty.

Our initial assumption is therefore untenable, and the proof of the theorem is complete.

Theorem 2. *Let n be a natural number, and suppose that $\mathcal{P} = E_1 \cup E_2 \cup E_3$, where every x-line, with at most a set of measure zero of exceptions, intersects E_1 in at most n points, and every t-line, with at most a set of measure zero of exceptions, intersects E_2 in a linear nowhere dense set of points. Then there exists a set of y-lines that is everywhere of positive exterior measure, every member of which intersects E_3 in a linear set of positive exterior measure.*

Proof. In the proof of Theorem 1, replace "first category" by "measure zero" and "(B_K)" by "(B_M)".

Theorem 3. *Let n be a natural number, and suppose that $\mathcal{P} = E_1 \cup E_2 \cup E_3$, where every x-line, with less than $2^{\aleph_0}$ exceptions, intersects E_1 in at most n points, and every t-line, with less than $2^{\aleph_0}$ exceptions, intersects E_2 in a linear nowhere dense set of points. Then there exists a set of y-lines that is everywhere of power $2^{\aleph_0}$, every member of which intersects E_3 in a set of power $2^{\aleph_0}$.*

Proof. In the proof of Theorem 1, replace "first category" by "power less $2^{\aleph_0}$" and "(B_K)" by "(B_P)".

Remark. Theorems 1, 2, and 3 are formally analogous to a theorem of ours ([2], Theorem 1) concerning the decomposition of three-dimensional space into three sets.

Corollary 1. *If n is a natural number, then $\mathcal{P}$ is not the union of three sets E_1, E_2, E_3 such that every x-line intersects E_1 in at most n points, every t-line intersects E_2 in a linear nowhere dense set of points, and every y-line intersects E_3 in a linear nowhere dense set of points.*

Proof. This corollary follows immediately from Theorem 1 and the fact that a nowhere dense set of points is not of second category.

Corollary 2. *If n is a natural number, then $\mathcal{P}$ is not the union of three sets E_1, E_2, E_3 such that every x-line intersects E_1 in at most n points, every t-line intersects E_2 in only a finite number of points, and every y-line intersects E_3 in less than $2^{\aleph_0}$ points.*

Proof. This corollary follows immediately from Theorem 3 and the fact that a finite set of points is nowhere dense.

Remark. Corollary 2 is formally analogous to a theorem due to Sierpiński ([6], Theorem 6) concerning the decomposition of three-dimensional space into three sets.

§ 2. Decompositions related to the continuum hypothesis.

For the proofs of the theorems to come, we require a subset of $\mathcal{E}$ with some of the properties of the set S of rational numbers used in the proof of Theorem 1, but having power $\aleph_1$ instead of $\aleph_0$. The existence of such a set is furnished by the following lemma.

Lemma 1. *There exists an everywhere dense set S of $\aleph_1$ real numbers such that, if $a \in S$ and $b \in S$, then $a + b \in S$ and $a - b \in S$.*

Proof. Let S_0 denote the set of rational numbers. Suppose that $0 < \eta < \omega_1$, and that an enumerable set S_ξ of real numbers has been defined for every $\xi < \eta$, with the properties that $S_\varrho \subset S_\xi$ for every $\varrho < \xi$ and that $a + b \in S_\xi$ and $a - b \in S_\xi$ if $a \in S_\xi$ and $b \in S_\xi$. Then

$$T_\eta = \bigcup_{\xi < \eta} S_\xi$$

is an enumerable subset of $\mathcal{E}$, and hence there exists a number

$$c_\eta \in \mathcal{E} - T_\eta.$$

Define S_η to be the set of all real numbers of the form

$$\pm a_1 \pm a_2 \pm \cdots \pm a_n \; (a_1, \ldots, a_n \in T_\eta \cup \{c_\eta\};\ n \text{ natural}).$$

Since T_η is enumerable, S_η is enumerable, and clearly $S_\varrho \subset S_\eta$ for every $\varrho < \eta$. Furthermore, it is evident, from the definition of S_η, that $a + b \in S_\eta$ and $a - b \in S_\eta$ if $a \in S_\eta$ and $b \in S_\eta$. The sets $S_\mu (\mu < \omega_1)$ are thus defined by transfinite induction. Let

$$S = \bigcup_{\mu < \omega_1} S_\mu.$$

Then, since $S_0 \subset S$, S is an everywhere dense subset of $\mathcal{E}$. Every $S_\mu (\mu < \omega_1)$ is enumerable, and hence S does not contain more than $\aleph_1$ elements. But S contains the $\aleph_1$ distinct numbers $c_\mu (\mu < \omega_1)$, and therefore S is of power $\aleph_1$. Suppose that $a \in S$ and $b \in S$. Then, since $\alpha < \beta < \omega_1$ implies that $S_\alpha \subset S_\beta$, there exists an $\eta < \omega_1$ such that $a \in S_\eta$ and $b \in S_\eta$. Consequently, $a + b \in S_\eta \subset S$ and $a - b \in S_\eta \subset S$, and the proof of the lemma is complete.

The theorems in § 1 make it natural to ask whether the following three propositions are true or false:

($\mathfrak{X}_K$) $\mathcal{P} = E_1 \cup E_2 \cup E_3$, *where every x-line, with at most a set of first category of exceptions, intersects E_1 in at most $\aleph_0$ points, every t-line, with at most a set of first category of exceptions, intersects E_2 in a linear nowhere dense set of points, and every y-line, with at most a set of first category of exceptions, intersects E_3 in a linear set of first category.*

($\mathfrak{X}_M$) $\mathcal{P} = E_1 \cup E_2 \cup E_3$, *where every x-line, with at most a set of measure zero of exceptions, intersects E_1 in at most $\aleph_0$ points, every t-line, with at most a set of measure zero of exceptions, intersects E_2 in a linear nowhere dense set of points, and every y-line, with at most a set of measure zero of exceptions, intersects E_3 in a linear set of measure zero.*

($\mathfrak{X}_P$) $\mathcal{P} = E_1 \cup E_2 \cup E_3$, *where every x-line, with less than* $2^{\aleph_0}$ *exceptions, intersects* E_1 *in at most* $\aleph_0$ *points, every t-line, with less than* $2^{\aleph_0}$ *exceptions, intersects* E_2 *in a linear nowhere dense set of points, and every y-line, with less than* $2^{\aleph_0}$ *exceptions, intersects* E_3 *in less than* $2^{\aleph_0}$ *points.*

The first two of the following three propositions were first discussed in [3]:

($\mathfrak{K}$) *The union of less than* $2^{\aleph_0}$ *subsets of* $\mathcal{S}$ *of first category is of first category.*

($\mathfrak{M}$) *The union of less than* $2^{\aleph_0}$ *subsets of* $\mathcal{S}$ *of measure zero is of measure zero.*

($\mathfrak{R}$) $2^{\aleph_0}$ *is a regular cardinal number.*

We shall show that the foregoing six propositions are related to the continuum hypothesis:

(H) $2^{\aleph_0} = \aleph_1$.

This will involve the following propositions, the first two of which were introduced in [4]:

($\mathfrak{B}_K^*$) *Let* $S < \mathcal{S}$, $T < \mathcal{S}$, S *be of power less than* $2^{\aleph_0}$ *and* T *be of first category. Then there exists an* $r \in \mathcal{S}$ *such that* $S[r] \cap T$ *is empty.*

($\mathfrak{B}_M^*$) *Let* $S < \mathcal{S}$, $T < \mathcal{S}$, S *be of power less than* $2^{\aleph_0}$ *and* T *be of measure zero. Then there exists an* $r \in \mathcal{S}$ *such that* $S[r] \cap T$ *is empty.*

($\mathfrak{B}_P^*$) *Let* $S < \mathcal{S}$, $T < \mathcal{S}$, S *be of power less than* $2^{\aleph_0}$ *and* T *be of power less than* $2^{\aleph_0}$. *Then there exists an* $r \in \mathcal{S}$ *such that* $S[r] \cap T$ *is empty.*

We shall require the following result concerning these three propositions.

Lemma 2. ($\mathfrak{K}$) *implies* ($\mathfrak{B}_K^*$), ($\mathfrak{M}$) *implies* ($\mathfrak{B}_M^*$), ($\mathfrak{R}$) *implies* ($\mathfrak{B}_P^*$).

As was pointed out, in connection with the first two of these implications, in [4], p. 84, the truth of Lemma 2 is apparent from the proofs of Theorems (B_K), (B_M), and (B_P).

Theorem 4. *The conjunction of* ($\mathfrak{K}$) *and* ($\mathfrak{X}_K$) *is equivalent to* (*H*).

Proof. (a) It is clear that (H) implies ($\mathfrak{K}$). According to [5], p. 9, (H) implies that $\mathcal{P} = E_1 \cup E_3$, where every x-line intersects E_1 in at most $\aleph_0$ points, and every y-line intersects E_3 in at most $\aleph_0$ points and hence in a linear set of first category. If we now take E_2 to be the empty set, we see that ($\mathfrak{X}_K$) is true.

(b) Assume that ($\mathfrak{K}$) and ($\mathfrak{X}_K$) are true, but that (H) is false. Then, according to ($\mathfrak{X}_K$), there exists a subset T of $\mathcal{S}$ of first category such that, for every $j \in \mathcal{S} - T$, the line $x = j$ intersects E_3 in a linear set of first category. Let S be a set satisfying the conditions stated in Lemma 1. According to Lemma 2, ($\mathfrak{B}_K^*$) holds, and hence there exists a real number h such that $S[h] \cap T$ is empty. Write $U = S[h]$. Then U is a subset of $\mathcal{S}$ of power $\aleph_1$, with the property that the union of the vertical lines $x = u (u \in U)$ intersects E_3 in a set of points whose projection on the y-axis is, because of ($\mathfrak{K}$), a linear set M of first category. Let the ordinates of the points of M constitute the subset C of $\mathcal{S}$. Define Q to be the set of real numbers q with the property that the t-line containing the point on the y-axis whose ordinate is q does not intersect E_2 in a linear nowhere dense set of points. The assertion in ($\mathfrak{X}_K$) concerning E_2 implies that Q is a subset of $\mathcal{S}$ of first category. It follows from ($\mathfrak{K}$) that the set

$$D = \bigcup_{u \in U} Q[u]$$

is a subset of $\mathcal{S}$ of first category. Let F be the set of real numbers f such that the line $y = f$ intersects E_1 in more than $\aleph_0$ points. The assertion in $(\mathfrak{X}_K)$ concerning E_1 implies that F is a subset of $\mathcal{S}$ of first category. Evidently the set

$$T' = C \cup D \cup F$$

is also a subset of $\mathcal{S}$ of first category. Applying $(\mathfrak{B}_K^*)$ again, we infer the existence of a real number h' such that $S[h'] \cap T'$ is empty. Write $V = S[h']$, and define Z to be the set of all points (u, v) of $\mathcal{P}$ with both $u \in U$ and $v \in V$. Since $V \cap C$ is empty, we have

$$Z < E_1 \cup E_2.$$

It follows now, just as in the proof of Theorem 1, that every t-line containing a point of Z not only intersects E_2 in a linear nowhere dense set of points, but, bearing in mind that S satisfies Lemma 1, also intersects every horizontal line $y = v$, with $v \in V$, in a point of Z.

Let V_0 be an enumerable, everywhere dense subset of V. For a fixed $\bar{v} \in V_0$, let $L_u (u \in U)$ denote the t-line that contains the point $(u, \bar{v})$ of Z. Since $V \cap F$ is empty, the assertion in $(\mathfrak{X}_K)$ concerning E_1 implies that, for every $v \in V_0$, at most enumerably many t-lines L_u, with $u \in U$, intersect the horizontal line $y = v$ in points of E_1. There are only $\aleph_0$ elements in V_0, but $\aleph_1$ elements in U, and hence there exists a t-line L_u with $u \in U$ such that L_u intersects every horizontal line $y = v$, with $v \in V_0$, in a point of E_2. But then the definition of V_0 implies that the points of E_2 are everywhere dense on this t-line that contains a point of Z, which is impossible.

Therefore (H) is true if $(\mathfrak{K})$ and $(\mathfrak{X}_K)$ are true.

Theorem 5. *The conjunction of* $(\mathfrak{M})$ *and* $(\mathfrak{X}_M)$ *is equivalent to* (H).

Proof. In the proof of Theorem 4, replace "$(\mathfrak{K})$" by "$(\mathfrak{M})$", "first category" by "measure zero", "$(\mathfrak{X}_K)$", by "$(\mathfrak{X}_M)$", and "$(\mathfrak{B}_K^*)$" by "$(\mathfrak{B}_M^*)$".

Theorem 6. *The conjunction of* $(\mathfrak{R})$ *and* $(\mathfrak{X}_P)$ *is equivalent to* (H).

Proof. In the proof of Theorem 4, replace "$(\mathfrak{K})$" by "$(\mathfrak{R})$", "first category" by "power less than $2^{\aleph_0}$", "$(\mathfrak{X}_K)$" by "$(\mathfrak{X}_P)$", and "$(\mathfrak{B}_K^*)$" by "$(\mathfrak{B}_P^*)$".

Even if the continuum hypothesis is assumed to hold, it is not known whether the following proposition is true or false:

(Σ) $\mathcal{P} = E_1 \cup E_2 \cup E_3$, *where every x-line intersects E_1 in a finite number of points, every t-line intersects E_2 in a finite number of points, and every y-line intersects E_3 in a finite number of points.*

The question of the truth or falsity of this proposition is very closely allied to a question raised by Sierpiński ([7], p. 399). A formal analogue of (Σ) concerned with three-dimensional space instead of the plane is known to be equivalent to the continuum hypothesis (see [7], p. 397). The following corollary shows that, under the assumption that $2^{\aleph_0}$ is regular, (Σ) implies that $2^{\aleph_0} = \aleph_1$.

Corollary 3. *The conjunction of* $(\mathfrak{R})$ *and* (Σ) *implies* (H).

Proof. It suffices to note that (Σ) evidently implies $(\mathfrak{X}_P)$, and then to apply Theorem 6.

Remark. We have shown elsewhere that $(\mathfrak{K})$ implies $(\mathfrak{R})$ ([3], p. 108, (VI. 1)), and that $(\mathfrak{M})$ implies $(\mathfrak{R})$ ([3], p. 110, VIII. 1). Hence, the conjunction of $(\mathfrak{K})$ and (Σ) implies (H), and the conjunction of $(\mathfrak{M})$ and (Σ) implies (H). This can be seen directly too from Theorems 4 and 5.

Bibliography.

[1] *F. Bagemihl*, A note on Scheeffer's theorem, Michigan Math. J. **2** (1953/54), 149—150.

[2] *F. Bagemihl*, Concerning the continuum hypothesis and rectilinear sections of spatial sets, Archiv d. Math. **10** (1959), 360—362.

[3] *F. Bagemihl*, Some results connected with the continuum hypothesis, Zeitschr. f. math. Logik u. Grundlagen d. Math. **5** (1959), 97—116.

[4] *F. Bagemihl*, Some propositions equivalent to the continuum hypothesis, Bull. Amer. Math. Soc. **65** (1959), 84—88.

[5] *W. Sierpiński*, Hypothèse du continu, Warszawa 1934.

[6] *W. Sierpiński*, Sur quelques propositions concernant la puissance du continu, Fund. Math. **38** (1951), 1—13.

[7] *W. Sierpiński*, Cardinal and ordinal numbers, Warszawa 1958.

Eingegangen 30. November 1959.

Urteile A. L. Crelles über seine Autoren.

Von *Kurt-R. Biermann* in Berlin.

Als Kurt Hensel aus Anlaß des 100jährigen Bestehens dieses Journals A. L. Crelles gedachte, führte er aus, daß jener „im höchsten Maße die Gabe" besaß, „fremdes wissenschaftliches Verdienst zu sehen und richtig zu bewerten." Und an Crelles Nachfolger C. W. Borchardt hob er hervor, daß er „mit Crelle die Kunst der Menschenbeurteilung und die Divinationsgabe für werdende große Talente gemein" gehabt habe[1]).

Als Hensel diese Zeilen schrieb, waren schon einige Dokumente, in denen sich Crelles Beurteilungskraft hinsichtlich fremden Könnens manifestierte, veröffentlicht; so war sein Eintreten für J. Steiner[2]) und für C. G. J. Jacobi[3]) durch die Publikation von Gutachten und Gesuchen Crelles belegt worden. Später hat Wilhelm Lorey die Eingaben Crelles zu Gunsten N. H. Abels abgedruckt[4]). In jüngster Zeit wurden vom Verf. weitere Belege G. Eisensteins[5]) und Lejeune Dirichlet[6]) betreffend publiziert, die der Urteilsgabe Crelles ein neues glänzendes Zeugnis ausstellen. Auch das Zusammenwirken Crelles mit Alexander von Humboldt sowohl bei der Förderung junger mathematischer Talente als auch bei den Bemühungen, den Fortbestand des Journals zu sichern, hat neue Darstellungen gefunden[7]).

Es sei nun hier der 180. Geburtstag August Leopold Crelles am 11. März 1960 zum Anlaß genommen, um einige weitere Ausführungen aus der Feder Crelles wiederzugeben, die unsere Kenntnis von seiner Feinfühligkeit für fremdes mathematisches Können vertiefen bzw. erweitern.

Bekanntlich hat Crelle Gründung und Existenz seines Journals unter persönlichen Opfern und nur durch immer wiederholte Bitten um Unterstützung durch die Behörden ermöglichen können[8]). In seinen Eingaben von 1827 bis 1854, die ihm in der Regel jeweils

---

[1]) Crelles Journal **157** (1927), 1—2.

[2]) *Julius Lange*, Jacob Steiners Lebensjahre in Berlin 1821—1863, Berlin 1899, 47—49 (Festschrift zur Erinnerung an das 75jährige Bestehen der Friedrichs-Werderschen Oberrealschule).

[3]) *Leo Koenigsberger*, Carl Gustav Jacob Jacobi, Leipzig 1904, 100—101 (Festschrift zur Feier der hundertsten Wiederkehr seines Geburtstage)s.

[4]) *Wilhelm Lorey*, Abels Berufung nach Berlin; Norsk Matematisk Tidskrift **11** (1929), 2—13. — Auszüge vorher schon bei Leo Koenigsberger, a. a. O.

[5]) *Kurt-R. Biermann*, A. L. Crelles Verhältnis zu Gotthold Eisenstein; Mon.Ber. Dt. Akad. Wiss. **1** (1959), 67—72.

[6]) *Kurt-R. Biermann*, Joh. Pet. Gust. Lejeune Dirichlet. Dokumente für sein Leben und Wirken, Berlin 1959, 44—49 (Abhh. Dt. Akad. Wiss., Kl. f. Math., Phys. u. Techn., 1959, Nr. 2).

[7]) Loc. cit. [6]), S. 49. — *Kurt-R. Biermann*, Über die Förderung deutscher Mathematiker durch Alexander von Humboldt; in: Alexander von Humboldt. Gedenkschrift zur 100. Wiederkehr seines Todestages, hrsg. v. d. Al. v. Humboldt-Komm. Dt. Akad. Wiss., Berlin 1959, 83—159.

[8]) *Wilhelm Lorey*, August Leopold Crelle zum Gedächtnis; Crelles Journal **157** (1927), 3—11.
Otto Emersleben, August Leopold Crelle (1780—1855) zum 100. Todestag; Wiss. Annalen **4** (1955), 651—656.

200 Taler einbrachten, hat nun Crelle stets eine genaue, von sichtlichem Stolz getragene Schilderung der Entwicklung sowohl des mathematischen als auch seines Journals für Baukunst[9]) gegeben, wobei er meist auch auf seine Autoren einging. Dabei ist es für unser Thema von Interesse, welche Mitarbeiter er besonders hervorhob, und daß er auch diese Gelegenheit nicht vorübergehen ließ, ohne hin und wieder zugunsten einzelner Mathematiker zu intervenieren, auch wenn das mit dem eigentlichen Zwecke seines Gesuches wenig zu tun hatte. So begegnen wir in den nachstehenden Auszügen aus Eingaben an Friedrich Wilhelm III. bzw. an dessen Nachfolger Friedrich Wilhelm IV. neben warmen Worten für Eisenstein, die das schon vorliegende Material zu dieser Frage ergänzen, u. a. auch einer Fürsprache für Weierstraß, in der Crelle dessen Bedeutung einen prophetischen Ausdruck gibt und die uns zeigt, daß Crelle der Weierstraßschen Berufung nach Berlin, die 1856 auf Kummers Veranlassung[10]) erfolgt ist, vorgearbeitet hat. Was die genannten Hervorhebungen von Mitarbeitern seines Journals betrifft, so finden wir freilich auch einige Namen mehr temporärer Bedeutung, es überwiegen aber die Nennungen von Ersten des Faches und eben darin offenbart sich wieder Crelles Einsicht und Weitblick.

Lassen wir nun Crelle selbst zu Wort kommen[11]):

3. 12. 1833 [12])

„. . . Die Theilnahme der Mathematiker vom Fach, und Anderer, die sich mit der höheren Mathematik beschäftigen, nimmt, wie oben angedeutet, für das Journal der Mathematik noch stets zu, und die Mitwirkung, die diese Schrift auf die Förderung der gegenwärtig so auffallend raschen weiteren Entwickelung dieser Wissenschaft ausübt, wird immer deutlicher erkennbar, und auch von den Mathematikern immer mehr anerkannt. In den letzten Jahren haben, um nur Einiges von dem Vielen zu nennen, was der Fortentwicklung der Mathematik förderlich gewesen ist, die Arbeiten von Abel, Jacobi, Dirichlet etc. der Analysis, und diejenigen von Steiner, Plücker, Möbius etc. der Geometrie wesentliche Erweiterungen verschafft . . .“

3. 12. 1841 [13])

„. . . Ungeachtet der Concurrenz sind [dem Journal] die Meister der Wissenschaft, selbst deren mehrere im Auslande, nicht ungetreu geworden. Mit Freuden aber darf man sagen, daß von jenen Geometern ersten Ranges jetzt mehrere dem *Inlande* angehören, wie zum Beispiel die Herren Jacobi, Dirichlet, Steiner und Andere. . . . Diese sind fortwährend Mit-Arbeiter an dem hiesigen Journal, während mancher berühmte Name des Auslandes, wie z. B. Gauß in Göttingen, Möbius in Leipzig, Plana in Turin, Hill in Lund und vieler Anderen dasselbe ebenfalls unter denen der Mit-Arbeiter schmückt. . . .“

---

[9]) Erschien seit 1829 und stellte 1851 mit dem 30. Bande sein Erscheinen ein.

[10]) *Kurt-R. Biermann, H. Grell, H. Reichardt* und *K. Schröder*, Beiträge zur Geschichte mathematischer, Lehre und Forschung an der Berliner Universität, Ms.

[11]) Es wird zitiert nach der im Deutschen Zentralarchiv, Abt. Merseburg, aufbewahrten Akte Rep. 89 B. XIV, 9, 10.

Diese Akte entstammt dem ehemaligen Brandenburg-Preußischen Hausarchiv. Darüber hinaus sind jedoch Crelles Journal betreffende Archivalien auch im ehemaligen Preußischen Geheimen Staatsarchiv aufbewahrt worden, welche aber noch nicht wieder aufgefunden werden konnten.

[12]) A. a. O., Bl. 20 R.

[13]) A. a. O., Bl. 49 R.

29. 1. 1844 [14])

„. . . Die namhaftesten Mathematiker im In- und Auslande, z. B. Jacobi, Dirichlet, Steiner, Kummer, Richelot, Möbius, Stern, Hill, Raabe etc. haben fortgefahren, ihre Arbeiten und Forschungen dem Journale zu überliefern; in der letzteren Zeit hat auch noch in Italien die Schrift mehrere namhafte Mit-Arbeiter sich gewonnen. . . .“

10. 7. 1845 [15])

„. . . Den immer neu aufsprießenden Talenten für diese Wissenschaft hat das Journal häufig gedient, ihre Forschungen bekannt zu machen. . . . Besonders hat das Journal an einem hiesigen jungen Mathematiker, dem Herrn Dr. Eisenstein, der sich, in noch jugendlichem Alter, schon durch die tiefsten Forschungen und durch wahre Entdeckungen, besonders in der Theorie der Zahlen, in hohem Grade ausgezeichnet hat, . . . einen trefflichen neuen Mitarbeiter erhalten, der noch viel Wichtiges in seiner Wissenschaft zu leisten verspricht. So, wie das Journal, vor nun beinahe 20 Jahren, das Glück hatte, die ersten und die meisten, jetzt von allen Kennern bewunderten Arbeiten des unvergeßlichen, leider zu früh dahin geschiedenen Abel zuerst bekannt zu machen, und dadurch diese classischen Arbeiten gleichsam begründet zu haben, so ist ihm jetzt wieder die Gelegenheit geworden, auch die ersten Arbeiten des Herrn Eisenstein, eines gleichfalls die Hoffnung auf Außerordentliches gebenden jungen Mathematikers, zuerst ins Publikum zu bringen. . . .“

16. 6. 1846 [16])

„. . . Die vollendeten neuen Bände enthalten wieder eine Menge der trefflichsten Arbeiten, z. B. von Jacobi, Steiner, Minding, Richelot usw. Auch durch das seltene Talent des Herrn Eisenstein, . . . welches sich schon so erfreulich bemerklich gemacht hat, sind ebenfalls die neueren Bände ferner bereichert worden. . . .“

6. 7. 1847 [17])

„. . . Die wieder erschienenen drei neuen Bände enthalten wieder bedeutende Arbeiten, z. B. von Jacobi, Dirichlet, Steiner, Plücker, Eisenstein, Richelot, welche, nächst vielen anderen, wohl noch in fernen Zeiten Werth für die Mathematik haben werden und wesentlich zu ihrer Erweiterung und fortgehenden Entwickelung beitragen. . . .“

26. 7. 1853 [18])

„. . . Wenngleich [dem Journal], leider, in den letzten Jahren die sehr thätigen Mitarbeiter Jacobi, Gudermann, Eisenstein und Göpel durch den Tod entzogen sind, bleiben ihm dennoch nicht allein eine große Zahl älterer namhafter Mitarbeiter, sondern es finden sich auch immer wieder neue bedeutende Theilnehmer. Auch die drei neuen Bände enthalten wieder treffliche Arbeiten ausgezeichneter Mathematiker, zum Beispiel von Richelot, Steiner, Graßmann, Kummer, Hesse, Schellbach, Möbius, Minding, Steichen, Oltramare, Raabe, Winkler etc., die an Gediegenheit den früheren nicht nachstehen. . . .“

[14]) A. a. O., Bl. 75 R.
[15]) A. a. O., Bl. 78 R u. 79 V.
[16]) A. a. O., Bl. 84 V.
[17]) A. a. O., Bl. 91 R.
[18]) A. a. O., Bl. 113 R.

27. 11. 1854 [19])

„. . . Die drei neuen Bände enthalten wiederum bedeutende . . . Arbeiten, sowohl von älteren Mitarbeitern, wie z. B. Dirichlet, Steiner, Schellbach, Joachimsthal, Hermite, Raabe etc., als von neu hinzu gekommenen z. B. Heine, Borchardt, Lottner, Stader, Weierstraß, Meissel, Kirchhoff, Hoffmann, Spottiswoode etc. Auch unter diesen Neueren befinden sich wieder ausgezeichnete Mathematiker. Besonders zeigen sich in den Arbeiten des Herrn Weierstraß, der jetzt bei dem Gymnasio zu Braunsberg in Ostpreußen als Oberlehrer angestellt ist, wie es auch sofort anerkannt wurde, so tief eindringende mathematische Einsichten und eine solche Begabung zu Forschungen, die mit Erfolg die Wissenschaft erweitern, daß er gleichsam die Reihe der berühmten, zu früh dahingeschiedenen Forscher Abel, Jacobi und Eisenstein fortsetzt, und gewiß noch Vieles, sehr Bedeutendes liefern würde, wenn er mit seinem seltenen Talent in eine Stellung kommen sollte, wo er sich weniger entfernt vom litterarischen Verkehr befände und in welcher das Unterrichtgeben ihm mehr Muße zu seinen Forschungen übrig ließe. . . .“

Crelle erlebte die Berufung Weierstraß' nach Berlin nicht mehr; ein Jahr nach dieser, seiner letzten Eingabe verstarb er am 6. 10. 1855 [20]), schon längere Zeit schwer leidend und alle ihm verbliebenen Kräfte seiner Schöpfung, dem Journal für die reine und angewandte Mathematik widmend.

Zum Abschluß sei noch einer zeitgenössischen, wohl weithin unbekannten Schilderung Crelles durch einen Mathematiker Raum gegeben, die das ergänzt, was wir von ihm durch Abel gehört haben. Der Norweger Otto Aubert besuchte 1834 Deutschland und berichtete über sein Zusammentreffen mit Crelle wie folgt [21]):

„Crelle ist ein ziemlich großer Mann, 53 Jahre alt, aber ziemlich kränklich (in drei Jahren hat er keinen natürlichen Schlaf gehabt), ein gut gewachsener und schöner Mann, mit einem außerordentlich sympathischen Gesicht. Er hat mich sehr freundlich aufgenommen, sprach von Abel, . . . Crelle machte mir Komplimente wegen meiner Abhandlung im Journal [22]), „es war ein hübscher Aufsatz“, er hatte mir über neue Beiträge schreiben wollen, er bedauerte, daß ich die Mathematik beiseitegelegt hätte usw., kurz gesagt, er war sehr wohlwollend. Crelle hat mich gebeten, um drei Uhr zu ihm zu kommen, damit wir den täglichen Spaziergang mit Steiner und Dirichlet machen könnten. . . . Crelle hat mir Mascheronis Abhandlung „die Theorie des Zirkels“ [23]) geliehen, welche ich mit großem Interesse und großer Begierde studiert habe. Also um drei Uhr wieder zu Crelle. Da traf ich Steiner, Dirichlet kam nicht. . . .“

---

[19]) A. a. O., Bl. 116 R.

[20]) Lorey gibt in der in Fußnote [8]) genannten Abh., S. 11, versehentlich 1856 als Todesjahr an.

[21]) *Sofie Aubert Lindbæk*, Landflygtige, Kristiania 1910, Teil II, S. 7ff. Auch über das Zusammentreffen mit Steiner, Plücker und Möbius berichtet Aubert übrigens sehr natürlich und unterhaltend.

Die Übersetzung habe ich der Güte von Herrn Prof. Viggo Brun, Drøbak bei Oslo, zu verdanken.

[22]) Vgl. *Kurt-R. Biermann* und *Viggo Brun*, Eine Notiz N. H. Abels auf einem Manuskript Otto Auberts; Nordisk Matematisk Tidskrift **6** (1958), 84—86.

[23]) *Lorenzo Mascheroni*, La geometria del compasso, Pavia 1797; frz. Übersetzg. v. A. M. Carette (Géométrie du compas) Paris ²1828.

Auberts Besuch bei Crelle dürfte etwa 20 Jahre nach dem Entstehen des diesem Heft beigegebenen Bildes Crelles stattgefunden haben[24]).

Wenngleich Crelles eigene mathematischen Arbeiten keine bleibenden Spuren hinterlassen haben, so sind doch seine Verdienste um die Förderung der Mathematik, der er sich mit Begeisterung und selbstloser Opferfreudigkeit hingegeben hat, und ihrer Vertreter so erheblich und nachwirkend, daß ihm wohl ein Platz auch in der „Neuen Deutschen Biographie" hätte eingeräumt werden sollen[25]). Sein Andenken zur Wiederkehr seines Geburtstages erneut ins Gedächtnis zurückzurufen, war die Absicht dieser Zeilen.

---

[24]) Das Ölgemälde befindet sich im Bezirksheimatarchiv Berlin-Schöneberg.

Der Maler ist unbekannt.

Die Reproduktion besorgte die Landesbildstelle Berlin.

Es handelt sich um das einzige uns bekannte Bild Crelles, auf das der Verfasser von Frau I. Theurich, Berlin, freundlicherweise aufmerksam gemacht wurde. In Crelles Journal sind schon die Bilder *späterer* Herausgeber wiedergegeben worden; nun nachträglich auch ein Porträt seines *Gründers* zu publizieren, erschien als eine Ehrenpflicht.

[25]) In der NDB 3 (1957) fehlt Crelle, während für die ADB 4 (1876), 589—590, Cantor seine Biographie geschrieben hat.

Eingegangen 9. März 1960.

sem

ren
der
hrer
uen
kehr
len.

rich,
raus-
eine

aphie